INTERMEDIATE ALGEBRA

Karl Friedrich Gauss was born at Brunswick, Germany, in 1777 and died at Göttingen in 1855. Justly called the greatest mathematician of modern times, he was scarcely less famous as physicist and astronomer; in fact, he was director of the observatory and professor of astronomy at Göttingen during the last forty years of his life.

He wrote on almost every phase of mathematics, and in many subjects the modern development is largely due to his genius.

Gauss was the first to explain satisfactorily the nature of imaginary quantities and to put their use in solving equations on a systematic and scientific basis. He brought into general use the symbol i for the imaginary unit $\sqrt{-1}$.

INTERMEDIATE ALGEBRA

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PREFACE

The *Intermediate Algebra* is designed to follow the authors' *Elementary Algebra.* It meets the most exacting requirements of college entrance and other examination boards and of the syllabi of various states. The presentation of topics, therefore, follows the traditional order.

While recognizing the increased maturity of the pupils, the authors nevertheless maintain in this present text that simple and interesting form of presentation which characterizes the earlier book. For example, while the axioms and fundamental laws are used in the proofs of theorems, yet these are stated and the proofs are given in an informal manner that at once attracts and holds the pupils' attention and interest.

Following are some of the features which deserve particular mention:

1. The abundant *exercises*, both oral and written, throughout the book, especially in the review of the four fundamental operations, and in factoring and fractions.

2. The unusual attention given to the *applications* of each topic in the carefully graded problems, especially in those parts of the book where the treatment is more extended than that given in the *Elementary Algebra.*

3. The clear and *simple development* of determinants of the second and third orders, and their use in solving systems of linear equations and in testing the consistency of such systems.

4. The *chapter on the number system of algebra* preceding quadratics and giving a simple and clear presentation of the elementary treatment of complex numbers just where it is most needed.

5. The chapter on *graphic representation* of quadratics, following the ordinary treatment and giving the natural sequel to the graphic work in the earlier course.

6. The *unique and ample treatment of radical expressions* and the rich collection of problems in the solution of which radicals are applied.

7. The brief but adequate *exposition of logarithms* for the purpose of solving problems involving higher powers and roots, as, for example, the very practical problems of compound interest.

A briefer course may be made, if desired, by the omission of certain chapters or topics which have been marked with a star. These omissions will not interrupt the continuity of the work.

It is hoped that the *Intermediate Algebra* will be found a satisfactory sequel to the *Elementary Algebra* and that together they may serve to extend still further the *concrete study of algebra*, with which the authors have been identified.

H. E. S.
N. J. L.

APRIL, 1916.

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INTERMEDIATE ALGEBRA

CHAPTER I

DEFINITIONS AND FUNDAMENTAL LAWS

1. **Algebra and Arithmetic.** Algebra differs from Arithmetic in three important respects:

1. *Letters are used systematically to represent numbers.*
2. *The equation is used extensively in the solution of problems.*
3. *The number system is extended to include negative and imaginary numbers.*

2. **The operations** of *addition, subtraction, multiplication, division,* and *extraction of roots* are called *algebraic operations.*

3. **Addition.** Numbers which are to be added are called *addends,* and the result obtained by adding numbers is called their *sum.*

4. **Subtraction.** The process of finding one of two numbers when the other number and their sum are given is called *subtraction.* The given number is the *subtrahend;* the given sum, the *minuend;* and the number to be found is the *remainder.*

5. **Multiplication.** The result obtained by multiplying one number by another is called their *product.* The numbers multiplied together are the *factors* of the product.

6. **Division.** The process of finding one of two numbers when the other number and their product are given is called *division.* The given number is the *divisor;* the given product, the *dividend;* and the number to be found is the *quotient.*

7. **Symbols of Operation.** The symbols $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, are used with the same meaning as in arithmetic.

8. **Coefficient.** In the product of two factors, either factor is called the coefficient of the other.

Thus, in $6a$, 6 is the coefficient of a, and a is the coefficient of 6. In $2ax$, $2a$ is the coefficient of x, $2x$ is the coefficient of a, and 2 is the coefficient of ax.

9. **Exponent.** A number written a little above and to the right of another number is called an *exponent*.

If an exponent is a positive integer, it indicates how many times the number under it is used as a factor.

Thus, $a^3 = a \cdot a \cdot a$, and $x^4 = x \cdot x \cdot x \cdot x$.

10. **Powers.** The product obtained by multiplying a number by itself one or more times is called a *power* of the number.

11. **Base.** A number multiplied by itself to form a power is called the *base* of the power.

Thus, in x^4, x is the base, x^4 is the power of the base, and 4 is the exponent of the power.

12. **Roots.** One of the two equal factors of a number is a *square root* of the number; one of its three equal factors is a *cube root* of the number; and in general, one of the n equal factors of a number is an *nth root of the number*.

Thus, 2 is a square root of 4, and 3 is a cube root of 27.

13. **The Radical Sign.** The symbol $\sqrt{}$ is called the *radical sign*. When placed over a number, it indicates that a root of that number is to be taken. A small numeral placed in the radical sign is called the index of the root, and shows what root is to be taken.

Thus, $\sqrt[3]{8}$ means a cube root of 8, and $\sqrt[4]{16}$ means a fourth root of 16. That is, $\sqrt[3]{8} = 2$ and $\sqrt[4]{16} = 2$.

The square root is indicated by the radical sign without an index.

Thus, $\sqrt{4}$ means a square root of 4.

14. Rational and Irrational Roots. A root which can be expressed in the form of an integer, or as the quotient of two integers, is said to be *rational*, while one which cannot be so expressed is *irrational*.

Thus, $\sqrt[3]{8} = 2$, $\sqrt{a^2 + 2ab + b^2} = a + b$, and $\sqrt{\frac{9}{4}} = \frac{3}{2}$ are rational roots, while $\sqrt[3]{4}$ and $\sqrt{a^2 + ab + b^2}$ are irrational roots.

15. Rational and Irrational Expressions. An algebraic expression which involves a letter in an irrational root is said to be *irrational with respect to that letter*; otherwise the expression is *rational with respect to the letter.*

Thus, $a + b\sqrt{c}$ is rational with respect to a and b, and irrational with respect to c.

16. Signed Numbers. There are many situations in which we start at a certain point and measure in opposite directions. Thus, latitude is measured *north* and *south* from the equator; longitude is measured *east* and *west* from the meridian of Greenwich; temperature is measured *above* and *below* zero, etc.

In algebra we have the signed numbers which are directly applicable to such cases.

The signed numbers of algebra consist of the ordinary numbers of arithmetic, together with numbers *opposite* to these, which are called *negative* numbers. The numbers of arithmetic are called *positive* numbers.

A negative number is indicated by placing the sign $-$ before it. A number with no sign before it is positive, but sometimes the sign $+$ is used to point out especially that a number is positive.

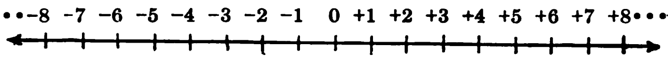
Thus, temperature is often designated as $+$ or $-$, according as it is above or below zero.

17. Absolute Value of Signed Numbers. The absolute value of a signed number is the value of the number when the sign is omitted.

Thus, the absolute value of $+3$, and also of -3, is 3.

18. Graphic Representation of Signed Numbers. One of the most extensive uses of signed numbers is for marking points on a straight line.

On an unlimited straight line mark one point zero. On both sides of it lay off equal divisions as shown in the figure. The points to the right of the zero point are marked $+$ and those to the left are marked $-$.



19. Addition of Signed Numbers. In addition, positive and negative numbers tend to cancel each other.

Thus, $+3+(-3)=0$, since -3 cancels all of $+3$, leaving zero.
Similarly, $+6+(-4)=+2$, since -4 cancels 4 out of 6, leaving $+2$.
Again, $+6+(-10)=-4$, since $+6$ cancels -6, leaving -4.

Signed numbers are added according to the following **rule**:

To add two numbers with like signs, find the sum of their absolute values, and prefix to this their common sign.

To add two numbers with opposite signs, find the difference of their absolute values, and prefix to this the sign of that one whose absolute value is the greater.

In case the signs of two numbers are opposite, and their absolute values are equal, their sum is zero.

20. Subtraction of Signed Numbers. The method of subtracting signed numbers is obtained directly from the definition of subtraction.

Thus, $8-(3)=5$, because $5+3=8$.
Similarly, $8-(-3)=11$, because $11+(-3)=8$,
$-8-(+3)=-11$, because $-11+3=-8$,
and $-8-(-3)=-5$, because $-5+(-3)=-8$.

These examples lead to the following **rule**:

To subtract one signed number from another signed number, change the sign of the subtrahend and add it to the minuend.

21. Multiplication of Signed Numbers. The rule for multiplying signed numbers may be found by considering the case in which the multiplier is an integer.

Thus, $3 \times 4 = 4 + 4 + 4 = 12,$

$3 \times (-4) = (-4) + (-4) + (-4) = -4 - 4 - 4 = -12,$

$-3 \times 4 = -(+4) - (+4) - (+4) = -4 - 4 - 4 = -12,$

and $-3 \times (-4) = -(-4) - (-4) - (-4) = +4 + 4 + 4 = +12.$

These examples lead to the following **rule**:

If two numbers have the same sign, their product is positive; if they have opposite signs, their product is negative.

The absolute value of the product is the product of the absolute values of the factors.

22. Division of Signed Numbers. The method of dividing signed numbers is obtained from the definition of division.

Thus, $8 \div 4 = 2$, because $2 \times 4 = 8$,

$8 \div (-4) = -2$, because $(-2) \times (-4) = 8$,

$-8 \div (-4) = 2$, because $2 \times (-4) = -8$,

and $-8 \div 4 = -2$, because $(-2) \times 4 = -8$.

Such examples lead to the following **rule**:

The quotient of two signed numbers is positive if the dividend and divisor have like signs; it is negative if they have opposite signs.

ORAL EXERCISES

Give the result in each of the following:

1. $8 + (-9)$	8. $-8 - 6$	15. $-8 \div 2$
2. $9 + (-3)$	9. $3 \times (-7)$	16. $-18 \div 6$
3. $7 - (-3)$	10. -6×4	17. $-21 \div (-7)$
4. $4 - (-8)$	11. $-3 \times (-9)$	18. $(-40) \div (-8)$
5. $3 - 8$	12. $-4 \times (-12)$	19. $(-36) \div 9$
6. $-3 - (-4)$	13. $12 \div (-6)$	20. $32 \div (-16)$
7. $-9 - (-3)$	14. $16 \div (-4)$	21. $-48 \div (-8)$

23. Fundamental Laws. While the operations of algebra are the same as those of arithmetic, they are sometimes carried out in a different manner, the difference being due largely to the representation of numbers by means of letters.

This leads to the study of certain *fundamental laws*, according to which these operations are carried out.

24. The Commutative Law of Addends. *Numbers to be added may be arranged in any order.*

Thus, $a + b + c = a + c + b = b + a + c$, etc.

25. The Associative Law of Addends. *Numbers to be added may be grouped in any manner.*

Thus, $a + b + c = a + (b + c)$.

The expression $a + b + c$ means that a is to be added to b, and then this sum is added to c, while $a + (b + c)$ means that b and c are added first and then a is added to the sum.

26. The Commutative Law of Factors. *Factors may be arranged in any order.*

Thus, $abc = acb = bac = bca$, etc.

27. The Associative Law of Factors. *Factors may be grouped in any manner.*

Thus, $abc = a(bc)$.

The expression abc means that b is multiplied by a, and c by this product; while $a(bc)$ means that c is first multiplied by b and this product is multiplied by a.

28. The Distributive Law of Multiplication with Respect to Addition. *The sum of two numbers may be multiplied by a third number by multiplying each number separately and adding the products.*

That is, $a(b + c) = ab + ac$.

If the equation $a(b + c) = ab + ac$ is read from right to left, it may be stated in words as follows:

To add numbers having a common factor, add the coefficients of the common factor and multiply the sum by the common factor.

29. If $a = \frac{1}{m}$, we have: $\frac{1}{m}(b+c) = \frac{1}{m}\cdot b + \frac{1}{m}\cdot c = \frac{b}{m} + \frac{c}{m}$, which is the distributive law of division with respect to addition.

This may be stated in words as follows:

To divide the sum of two numbers, divide each one separately and add the quotients.

If c is a negative number, $a(b+c) = ab + ac$ gives rules for multiplying or dividing the *difference of two numbers.*

30. The commutative and associative laws of factors give rise to the following **rule**:

To multiply or divide a product, multiply or divide any one of the factors, leaving the other factors unchanged.

Thus, $4 \cdot (2 \cdot 3) = (4 \cdot 2) \cdot 3 = 8 \cdot 3 = 24$; or $4 \cdot (2 \cdot 3) = (4 \cdot 3) \cdot 2 = 12 \cdot 2 = 24$. Also $\frac{2 \cdot 4 \cdot 6}{2} = 1 \cdot 4 \cdot 6 = 2 . 2 \cdot 6 = 2 \cdot 4 \cdot 3 = 24$.

31. Division by Zero. Using the signed numbers of algebra, we have:

1. Any two numbers have one and only one sum.
2. Any two numbers have one and only one difference.
3. Any two numbers have one and only one product.
4. Any number divided by any number gives one and only one quotient *provided the divisor is not zero.*

In case the divisor is zero, we need to consider two cases:

(*a*) If the dividend a is different from zero, then there is no number k such that

$$\frac{a}{0} = k,$$

for by the definition of division, $a = 0 \cdot k = 0$, which contradicts the provision that a is different from zero.

(*b*) If the dividend a is zero, then $\frac{a}{0}$ equals any number whatever. That is

$$\frac{0}{0} = k,$$

for any value of k, since $0 = 0 \cdot k$ for all values of k.

Hence *dividing a number different from zero by zero is an impossible operation. Dividing zero by zero leads to an indeterminate result. In no case, therefore, is division by zero permitted.*

32. Symbols of Aggregation. Parentheses are used to indicate that some operation is to be extended over the whole expression inclosed by them. Thus, $2(x+y)$ means that the sum of x and y is to be multiplied by 2, while $2x+y$ means that x alone is to be multiplied by 2.

Instead of parentheses, brackets [ ], braces { }, or the vinculum $\overline{\quad\quad}$, may be used with the same meaning. All such symbols are called symbols of aggregation.

Thus, $2(x+y)$, $2[x+y]$, $2\{x+y\}$, $2\,\overline{x+y}$ all mean the same thing.

In the sign $\sqrt{\quad}$, the horizontal bar is a sign of aggregation, indicating that the root of the whole expression under the bar is to be taken. Thus, $\sqrt{a+b}$ means the square root of the sum of a and b.

The dividing line of a fraction is a symbol of aggregation, indicating that the whole of the numerator is to be divided by the whole of the denominator.

33. Order of Operations. In a succession of indicated operations the final result depends in some cases upon the order in which the operations are performed, while in some cases it does not.

Thus, $36 \div 6 \div 3 = 6 \div 3 = 2$, if 36 is divided by 6 first and this result by 3. On the other hand, $36 \div 6 \div 3 = 36 \div 2 = 18$, if 6 is divided by 3 first, and then 36 divided by the quotient.

To prevent mistakes, and to make usage uniform, the following rules have been adopted:

In an expression involving additions, subtractions, multiplications, divisions, powers, and roots, when no symbols of aggregation are involved,

(1) *All powers and roots are found first.*

(2) *All multiplications are performed next, and these may be taken in any order.*

(3) *All divisions are performed next, and these are taken in the order in which they occur from left to right.*

(4) *Finally, additions and subtractions are performed, and these may be taken in any order.*

CHAPTER II

FUNDAMENTAL OPERATIONS

34. Algebraic Expressions. Any combination of numerals, letters, and signs of operation, used for the purpose of representing numbers, is called an *algebraic expression.*

35. Polynomials; Terms. An algebraic expression composed of parts connected by the signs $+$ and $-$ is called a **polynomial.** Each of the parts thus connected, together with the sign preceding it, is called a **term.**

E.g. $5a - 3xy - \frac{2}{3}rt + 99$ is a polynomial whose terms are $5a$, $-3xy$, $-\frac{2}{3}rt$, and $+99$. The sign $+$ is understood before $5a$.

A polynomial of two terms is called a **binomial**; one of three terms is called a **trinomial.** A term taken by itself is called a **monomial.**

E.g. $5a - 3xy$ is a binomial; $5a - 3xy - \frac{2}{3}rt$ is a trinomial whose terms are the monomials $5a$, $-3xy$, $-\frac{2}{3}rt$.

According to the above definition $a + (b + c)$ may be called a binomial though it is equivalent to the trinomial $a + b + c$.

In this case a is called a **simple** term and $(b + c)$ a **compound** term. Likewise we may call $3t + 4x - 5(a + b)$ a trinomial having the simple terms $3t$, $4x$, and the compound term $-5(a + b)$.

36. Similar Terms. Two terms which have a factor in common are said to be **similar with respect to that factor.**

E.g. $5a$ and $-3a$ are similar with respect to a; $-3xy$ and $-7x$ are similar with respect to x; $5a$ and $-5b$ are similar with respect to 5; $7abc$ and $-\frac{2}{3}abc$ are similar with respect to abc.

ADDITION AND SUBTRACTION OF MONOMIALS

37. Adding Similar Terms. In accordance with § 28, the sum of terms which are similar with respect to a common factor is equal to the product of this common factor and the sum of its coefficients.

Notice that this rule applies at once to subtraction because, by § 20, subtraction is replaced by the addition of the subtrahend with its sign changed.

Example 1. $8ax^2 + 9ax^2 - 3ax^2 = [8 + 9 + (-3)]\,ax^2 = 14ax^2.$

Example 2. $a\sqrt{x^2+y^2} + b\sqrt{x^2+y^2} = (a+b)\,\sqrt{x^2+y^2}.$

Example 3. $(x+3)(x-4) + (2x-5)(x-4) + (3x-2)(x+7).$

Adding the first two terms,

$$\begin{aligned}(x+3)(x-4)+(2x-5)(x-4) &= (x+3+2x-5)(x-4)\\ &= (3x-2)(x-4)\end{aligned}$$

Adding this sum to the third term,

$$\begin{aligned}(3x-2)(x-4)+(3x-2)(x+7) &= (x-4+x+7)(3x-2)\\ &= (2x+3)(3x-2),\end{aligned}$$

which is the required sum.

ORAL EXERCISES

Perform the following indicated operations:

1. $5bc + 3bc + 4bc - 8bc.$
2. $4b^2x + 6b^2x - 3b^2x - 2b^2x.$
3. $7b^5c^4 - 9b^5c^4 + 3b^5c^4 - 18b^5c^4.$
4. $11a^6x^4 - 15a^6x^4 + 24a^6x^4 - 3a^6x^4.$
5. $14a^3x^5 - 11a^3x^5 - 14a^3x^5 + 11a^3x^5.$
6. $ax^2 - bx^2 + cx^2 - dx^2.$
7. $5(n-1) + 3(n-1) - 6(n-1).$
8. $4(n+1) - 8(n+1) + 11(n+1).$
9. $a(n-1) + b(n-1) - c(n-1).$
10. $3n(n-1) + 2n(n-1) - 4n(n-1).$
11. $n(n-1) - 2(n-1) + 3(n-1).$

12. $n(n+1)-4(n+1)+3(n+1)$.

13. $3\sqrt{x+1}+6\sqrt{x+1}-5\sqrt{x+1}$.

14. $8\sqrt{x-1}+3\sqrt{x-1}-8\sqrt{x-1}$.

15. $3a^n-7a^n+15a^n-20a^n$.

16. $2a^nx+3a^nx-6a^nx+5a^nx$.

17. $2a^2b^n+3a^2b^n-5a^2b^n+6a^2b^n$.

18. $5x^5y^5-10x^5y^5+15x^5y^5-20x^5y^5$.

19. $3x^{n+3}+6x^{n+3}-9x^{n+3}+12x^{n+3}$.

20. $7x^{a+6}-14x^{a+6}+42x^{a+6}+63x^{a+6}$.

WRITTEN EXERCISES

1 $7x^{4a}-14x^{4a}-21x^{4a}+28x^{4a}$.

2. $ab^n+3ab^n+5ab^n-6ab^n$.

3. $12x^2y^5+10x^2y^5-3ax^2y^5+ax^2y^5$.

4. $a(b+3)-3a(b+3)+4a(b+3)+2a(b+3)$.

5. $(a+1)(b+3)+(a-1)(b+3)-a(b+3)$.

6. $(a+b)(x-y)+(a-b)(x-y)+b(x-y)$.

7. $(a+3)(x+y)+(a-4)(x+y)+(2-a)(x+y)$.

8. $n(n-1)(n-2)+(n-1)(n-2)$.

9. $n(n-1)+n(n-1)(n-2)=n(n-1)[1+n-2]$
$=n(n-1)(n-1)=n(n-1)^2$.

10. $n(n-1)(n-2)+n(n-1)(n-2)(n-3)$.

11. $n(n-1)(n-2)(n-3)+(n-1)(n-2)(n-3)$.

In the following add the first and second terms, then this result and the last term:

12. $(a-4)(b+3)+(a+3)(b+3)+(2a-1)(b-2)$.

13. $(x+2y)(x-2y)+(x-3y)(x-2y)+(2x-y)(x-y)$.

14. $(3x-a)(y-b)+(3x+a)(y-b)+6x(y+b)$.

15. $(n+1)(x+a)+(1-4n)(x+a)+(2-3n)(x-a)$.

16. $(n-1)(n-3)+2(n-3)+n(n+1)$.

17. $(2a-3)(3a-2)+(a+4)(3a-2)+(a-4)(3a+1)$.

ADDITION AND SUBTRACTION OF POLYNOMIALS

38. Adding Polynomials. In adding polynomials we make use of the associative and commutative laws (§§ 24, 25), in order to group the terms in a convenient manner.

Example. Add $5a^2 - 3ab - 2b^2$ and $3a^2 + 6ab - 4b^2$.

Arranging similar terms in columns, we have

$$\begin{array}{r} 5a^2 - 3ab - 2b^2 \\ \underline{3a^2 + 6ab - 4b^2} \\ 8a^2 + 3ab - 6b^2 \end{array}$$

39. Subtracting Polynomials. In like manner the terms are arranged in subtracting one polynomial from another.

Example. Subtract $4x - 2y + 6z$ from $3x + 6y - 3z$.

$$\begin{array}{r} 3x + 6y - 3z \\ \underline{4x - 2y + 6z} \\ -x + 8y - 9z \end{array}$$

The steps are:
$3x - 4x = -x$; $6y - (-2y) = 8y$; $-3z - (+6z) = -9z$.

Oral Addition of Polynomials. The sum of polynomials may also be found directly without arranging similar terms in columns as is shown in the following example:

Example. Add, $3x^3 + 7x^2 - 4x + 7$; $2x^3 + 8x - 9$; $5x^2 - 3x - 2$ and $5x^3 - 2x^2 + 7x - 3$.

Solution. We notice that the coefficients of terms containing x^3 are 3, 2, and 5. Hence the sum of these terms is $10x^3$. Similarly the sum of the terms containing x^2 is $10x^2$; the sum of the terms containing x is $8x$ and the sum of the remaining terms is -7. Hence the required sum is $10x^3 + 10x^2 + 8x - 7$.

ORAL EXERCISES

Add the following:

1. $5x^2 - 3x + 2$, $-4x^2 + 7x - 3$, $5x - 7$, $9x^2 - 3x$.

2. $2x^4 - 3x^3 - 4x^2 + 2x - 7$, $x^4 + 4x^3 - 7x^2 - 3x + 4$, $4x^3 + 8x^2 - 9$.

3. $9a^3 - 4a + 7a^2 - 3$, $5a^2 + 6a - 9$, $2a^2 - 4 + 5a$, $8a^3 - 3a^2 - 8 + 3a$.

4. Add $8x^3 - 11x - 7x^2$, $2x - 6x^2 + 10$, $-5 + 4x^3 + 9x$, and $13x^2 - 5 - 12x^3$.

5. Add $5a^3 - 2a - 12 - 10a^2$, $14 - 7a + a^2 - 9a^3$, $3a^2 - 13a^3 + 4 - 11a$, and $3 - 7a + 10a^2 + 4a^3$.

WRITTEN EXERCISES

1. Add $37a - 4b - 17c + 15d - 6f - 8h$ and $3c - 31a + 9b - 5d - h - 4f$.

2. Add $11q - 10p - 8n + 3m$, $24m - 17q + 15p - 13n$, $9n - 6m - 4q - 7p - 5n$, and $8q - 4p - 12m + 18n$.

3. From the sum of $9m^3 - 3m^2 + 4m - 7$ and $3m^2 - 4m^3 + 2m + 8$ subtract $4m^3 - 2m^2 - 4 + 8m$.

4. From the sum of $x^4 - ax^3 - a^2x^2 - a^3x + 2a^4$ and $3ax^3 + 7a^2x^2 - 5a^3x + 2a^4$ subtract $3x^4 + ax^3 - 3a^2x^2 + a^3x - a^4$.

5. From the sum of $13a - 15b - 7c - 11d$ and $7a - 6b + 8c + 3d$ subtract the sum of $6d - 5b - 7c + 2a$ and $5c - 10d - 28b + 17a$.

6. Add $(a + b - c)m + (a - b + c)n + (a - b - c)k$,
$(2a - 3b + c)m + (-3a + b + c)n + (a + 2b + 4c)k$,
and $(b - 2c)m + (2a + b - 2c)n + (-2a + 2b + c)k$.

7. From the sum of $ax^3 - bx^2 + cx - d$ and $bx^3 + ax^2 - dx + c$ subtract $(a - b)x^3 + (c - a)x^2 - (b + d)x - d + c$.

8. Add $(m^2 - 2mn + n^2)x^3 + (m^2 + 2mn + n^2)x^2 - (m + n)x + 8$,
$(m^2 + n^2)x^3 + (-2mn)x^2 + (m - n)x - 31$,
$(m^2 + mn + n^2)x^3 + (m^2 - n^2)x^2 - (2m + n)x + 4$.

9. Add $a^n + 2a^{n+1} + a^{n+2}$ and $2a^n - 4a^{n+1} + 5a^{n+2}$ and from this sum subtract $7a^{n+1} - 8a^n + a^{n+2}$.

10. Add $3x^a + 2x^{a-1} + a^{n-2}$ and $5x^a - 7x^{a-1} - 3a^{n-2}$, and from this sum subtract $-8x^a + 15x^{a-1} + 7a^{n-2}$.

REMOVAL OF PARENTHESES

40. Parentheses Preceded by a Plus Sign. Since $a+(+b)=a+b$ and $a+(-b)=a-b$, it follows that parentheses preceded by a plus sign may be removed without changing any sign within.

Thus, $a+(b-c+d-e)=a+b-c+d-e$.

41. Parentheses Preceded by a Minus Sign. Since $a-(+b)=a-b$, and $a-(-b)=a+b$, it follows that parentheses preceded by a minus sign may be removed by changing the sign of each term within.

Thus, $a-(b+c-d+e)=a-b-c+d-e$.

42. Parentheses within Parentheses. In case an expression contains signs of aggregation, one within another, these may be removed one at a time, beginning with the *innermost*, as in the following example:

$$\begin{aligned} & a-\{b+c-[d-e+f-(g-h)]\} \\ =\; & a-\{b+c-[d-e+f-g+h]\} \\ =\; & a-\{b+c-d+e-f+g-h\} \\ =\; & a-b-c+d-e+f-g+h. \end{aligned}$$

43. Such combinations of signs of aggregation may also be removed in order, beginning with the *outermost*, by observing *the number of minus signs which affect each term, and calling the sign of a term + if this number is even, − if this number is odd.*

Thus, in the above example, b and c are each affected by one minus sign, namely the one preceding the brace. Hence we write, $a-b-c$.

d and f are each affected by two minus signs, namely the one before the brace and the one before the bracket, while e is affected by these two and also by the one preceding it. Hence we write, $d-e+f$.

g is affected by the minus signs before the bracket, the brace, and the parenthesis, an *odd* number, while h is affected by these and also by the one preceding it, an *even* number. Hence we write, $-g+h$.

By counting in this manner as we proceed from left to right, we write the result in the final form at once, namely, $a-b-c+d-e+f-g+h$.

WRITTEN EXERCISES

In removing the signs of aggregation in the following, either of the two processes just explained may be used. The second method is shorter and should be easily followed after a little practice.

1. $7-\{-4-(4-[-7])-(5-[4-5]+2)\}$.
2. $-[-(7-\{-4+9\}-13)-(12-3+\overline{-7+2})]$.
3. $6-(-3-[-5+4]+\{7-3-\overline{7-19}\}+8)$.
4. $5+[-(-\{-5-3+11\}-15)-3]+8$.
5. $4x-[3x-y-\{3x-y-(x-\overline{y-x})+x\}-3y]$.
6. $3x^2-2y^2-(4x^2-\{3x^2-\overline{y^2-2x^2}-3y^2\}-y^2+4x^2)$.
7. $7a-\{3a-[-2a-\overline{a+3}+a]-\overline{2a-5}\}$.
8. $l-(-2m-n-\{l-m\})-(5l-2n-[-3m+n])$.
9. $2d-[3d+\{2d-(e-5d)\}-(d+3e)]$.
10. $4y-(-2y-[-3y-\{-y-\overline{y-1}\}+2y])$.
11. $3x-[8x-(x-3)-\{-2x+6-\overline{8x-1}\}]$.
12. $x-(x-\{-4x-[5x-\overline{2x-5}]-[-x-\overline{x-3}]\})$.
13. $3x-\{y-[3y+2z]-(4x-[2y-3z]-\overline{3y-2z})+4x\}$.
14. $x-(-x-\{-3x-[x-\overline{2x+5}]-4\}-[2x-\overline{x-3}])$.
15. $a-[5b-\{a-(5c-\overline{2c-b}-4b)+2a-(a-\overline{2b+c})\}]$.
16. $2(3b-5a)-7[a-6\{2-5(a-b)\}]$.
17. $-2\{a-6[a-(b-c)]\}+6\{b-(c+a)\}$.
18. $-3\{-2[-4(-a)]\}+5\{-2[-2(-a)]\}$.
19. $-2\{-[-(x-y)]\}+\{-2[-(x-y)]\}$.
20. $a-(b-c)-[a-b-c-2\{b+c-3(c-a)-d\}]$.
21. $2x-(3y-4z)-\{2x-[3y+4z-3y-(4z+2x)]\}$.
22. $-2(a-d)-2[b+c+d-3\{c+d-4(d-a)\}]$.
23. $-4(a+d)+4(b-c)-2[c+d+a-3\{d+a-4(b+c)\}]$.
24. $a-2b-[4a-6b-\{3a-c+(5a-2b-\overline{3a-c+2b})\}]$
25. $a-[-b\{-c(-d+\overline{e-f})+2a-c\}+c+d]$.

MULTIPLICATION OF MONOMIALS

44. Adding Exponents in Multiplication. In $2^2 \cdot 2^3 = (2 \cdot 2)(2 \cdot 2 \cdot 2) = 2^{2+3} = 2^5$, we see that the exponents of the factors are added to obtain the exponent of the product.

Similarly, we have in general,

$$b^k \cdot b^n = b^{k+n}.$$

For by the definition of a positive integral exponent,

$$b^k = b \cdot b \cdot b \cdots \text{to } k \text{ factors}$$

and $$b^n = b \cdot b \cdot b \cdots \text{to } n \text{ factors.}$$

Then, $$\begin{aligned} b^k \cdot b^n &= (b \cdot b \cdots \text{to } k \text{ factors})(b \cdot b \cdots \text{to } n \text{ factors}) \\ &= b \cdot b \cdot b \cdots \text{to } k + n \text{ factors.} \end{aligned}$$

Hence, $$b^k \cdot b^n = b^{k+n}.$$

That is: *The product of two powers of the same base is a power of that base whose exponent is the sum of the exponents of the common base.*

45. Multiplying Monomials. In finding the product of two monomials, the factors may be *arranged* and *grouped* in any manner, according to the commutative and associative laws (§§ 26, 27).

E.g. $$\begin{aligned} (3ab^2) \times (5a^2b^3) &= 3ab^2 \cdot 5a^2b^3 \\ &= 3 \cdot 5 \cdot a \cdot a^2 \cdot b^2 \cdot b^3 \\ &= (3 \cdot 5)(a \cdot a^2)(b^2 \cdot b^3) \\ &= 15a^3b^5. \end{aligned}$$

The factors in the product are arranged so as to associate those consisting of Arabic figures and also those which are powers of the same base.

Positive and Negative Products. It is readily seen that a product is negative when it contains an *odd* number of *negative* factors; otherwise it is positive.

For by the commutative and associative laws of factors the negative factors may be grouped in *pairs*, each pair giving a *positive* product. If the number of negative factors is odd, there will be just one remaining, after the others are paired and multiplied together, and this makes the final product negative.

ORAL EXERCISES

Find the products of the following:

1. $3x^2, 2x^3$.
2. $2a^2, 4a^4$.
3. $3y^3, 3y^3$.
4. $4a^5, 2a^4$.
5. $5a^4, 4a^5$.
6. $3a^4, 4a^3$.
7. $2a^5, 2a^6$.
8. x^n, x^n.
9. $2x^n, 3x^2$.
10. $3x^n, 3x^n$.
11. x^n, x^{n+1}.
12. x^{2n}, x^n.
13. $x^{2n}, 2x^{2n}$.
14. x^{n+1}, x^n.
15. x^{n+1}, x^{n+2}.
16. x^{n-1}, x^{n+1}.
17. x^{n-1}, x^{n+3}.
18. x^{1-n}, x^{1+n}.
19. $5x^{2n}, 2x^{3n}$.
20. $6x^{3n+1}, 2x^{n+3}$.
21. $7a^{2n+1}, 3a^{n+5}$.
22. $6x^{a+3}, 2x^{2a-2}$.
23. $5a^{2n-7}, 3a^{n+8}$.
24. $4a^{4n+2}, 4a^{2n-6}$.
25. $3y^{4a}, 7y^{5a+2}$.
26. $9x^{3b}, 2x^{b+a}$.
27. $4a^{2n-a}, 3a^n$.
28. $2x^{4a-n}, 2x^{a+n}$.
29. $4x^{7n-2a}, 5x^{3n+3a}$.
30. $8x^{3a-8n}, 2x^{4a+9n}$.

WRITTEN EXERCISES

Find the products of the following:

1. $2^3 \cdot 3^4, 2^7 \cdot 3^2$.
2. $2a^2b, 3ab^2, a^3b^3$.
3. $2x^2y^3, 5x^3y^2, 2x^4y$.
4. $5xy, 2x^3y, 4xy^5, x^2y^2$.
5. $3a^5bc, ab^2c, a^2bc^4, 4ab^5c$.
6. $x^n, x^{n-1}, x^{n+1}, 2x^n$.
7. $x^{m+n-1}, x^{m-n+1}, x^{2m}$.
8. a^x, a^{3x-y}, a^{y-3x}.
9. $a^nb^m, a^{2n}b^{3m}, a^{1-3n}b^{2-4m}$.
10. $4ab^m, 2a^3b^n, 3a^6b^{2-m-n}$.
11. $2x^my^{m+n}, 3x^{m-1}y^{2n-m+2}$.
12. $a^{d-2c+2}b^{m-3n}, a^{2c-d-1}b^{2-m+3n}$.
13. $3x^{a+3b}, 2x^{a-2b}y^{c-3}, 2x^{4-2a-b}y^{2c+3}$.
14. $a^{2x-3}b^{y+1}, a^{x+3}b^{y-1}, 3a^3b^2$.
15. $3x^my^n, 2x^ny^m$.
16. $x^{a+1}, x^{b-1}y^c, y^{1-c}$.
17. $7x^{3a-4}, 3x^{5-4a}, 2x^{1-a}$.
18. $2a^{m-n}, 3a^{2m+n}, 4a^{1-m}$.
19. $3^{2-5m+3n} \cdot 2^{4a-3b}, 3^{2-3n+6m} \cdot 2^{5+3b+5a}$.
20. $(1+a)^{7-3b+a} \cdot (1-a)^{2+a-b}, (1-a)^{b-a-1} \cdot (1+a)^{3b-a-6}$.
21. $(x-3)^{4n-5a+2}. \quad (x+3)^{2a-3n-1}, (x-3)^{6a-4n-2}. \quad (x+3)^{4n-3a+1}$.
22. $(a+b)^{a+b}. \quad (a-b)^{a-b}, (a+b)^{a-b}. \quad (a-b)^{a+b}$.
23. $(x+y+1)^{x+y+1}, (x-y+1)^4, (x+y+1)^{x-y-1}$.

DIVISION OF MONOMIALS

46. Subtracting Exponents in Division. We know that $x^6 \div x^4 = x^2$, because $x^2 \times x^4 = x^6$. That is, the quotient times the divisor equals the dividend. But $x^2 = x^{6-4}$. Therefore the exponent of the quotient is equal to *the exponent of the dividend minus that of the divisor.*

In general, if a is any number, and m and k are any positive integers, of which m is the greater, then

$$a^m \div a^k = a^{m-k}.$$

For, since k and $m-k$ are both positive integers, we have $a^k \times a^{m-k} = a^{k+m-k} = a^m$. That is, a^{m-k} is the number which multiplied by a^k gives a product a^m.

Hence we have the

Rule. *The quotient of two powers of the same base is a power of that base whose exponent is the exponent of the dividend minus that of the divisor.*

Negative and Zero Exponents. Under the proper interpretation of negative numbers used as exponents, this principle also holds when $m < k$. This is considered in detail in Chapter XII. We remark here that in case $m = k$, the dividend and the divisor are equal, and the quotient is unity. Hence,

$$a^m \div a^m = a^{m-m} = a^0 = 1.$$

47. Dividing Monomials. In dividing monomials, we make use of the commutative and associative laws in order to group together as far as possible powers having a common base.

Example 1. Divide $2^4\, x^5y^6$ by $2^2\, x^2y^4$.

Solution. We divide 2^4 by 2^2, x^5 by x^2, and y^6 by y^4, and multiply the quotients.

Thus, $$2^4\, x^5y^6 \div 2^2\, x^2y^4 = 2^{4-2}\, x^{5-2}y^{6-4} = 2^2\, x^3y^2.$$

Example 2. Divide $6\, b^4c^2x^3$ by $4\, bc^2x^2$.

Solution. $$6\, b^4c^2x^3 \div 4\, bc^2x^2 = \frac{6}{4} \cdot \frac{b^4}{b} \cdot \frac{c^2}{c^2} \cdot \frac{x^3}{x^2} = \frac{3}{2} b^3x.$$

ORAL EXERCISES

Find the quotients of the following:

1. $4x^4 \div 2x^2$.
2. $6x^5 \div 3x^2$.
3. $8x^6 \div 2x^5$.
4. $8x^5 \div x^4$.
5. $10x^7 \div x^5$.
6. $-24ax^2 \div 4ax$.
7. $15x^{2n} \div 3x^n$.
8. $15x^{3n} \div 5x^{2n}$.
9. $-24x^{4a} \div 12x^{2a}$.
10. $16x^{6n} \div (-8x^{2n})$.
11. $14x^{5n} \div 7x^{4n}$.
12. $-18a^{2n+1} \div (-6a^n)$.
13. $32a^{3n+2} \div 16a^{n+2}$.
14. $-6x^{4n+3} \div x^{3n+2}$.
15. $10a^{5n+3} \div 2a^{2n+1}$.
16. $12a^{4n+7} \div (-4a^{n+6})$.
17. $a^{a-b} \div a^{b-a}$.
18. $x^{3n+2} \div x^{2n+1}$.
19. $4x^{5n+4} \div 2x^{5n-5}$.
20. $10x^{4n+5} \div 5x^{4n-6}$.
21. $12x^{7n+3} \div 6x^{7n+2}$.
22. $x^{x+y} \div x^{x-y}$.
23. $24a^{2n+3} \div 8a^{n+2}$.
24. $32a^{3n-2} \div 8a^{2n-4}$.

WRITTEN EXERCISES

Divide:

1. $2^4 \cdot 3^7 \cdot 5^2$ by $2^3 \cdot 3^4 \cdot 5$.
2. $3^7 \cdot 7^4 \cdot 13^5$ by $3^5 \cdot 7^2 \cdot 13^2$.
3. $3x^7y^2z$ by $2x^3yz$.
4. $5a^5b^7c^8$ by $5a^4b^7c^4d^2$.
5. $x^{2n}y^mz^{3m}$ by $x^ny^mz^m$.
6. $a^{3n-5}y^{2n+3}$ by $a^{n+6}y^{2n+1}$.
7. $a^{c+3d+2}b^{d-2c+6}$ by a^{c+2d-4}.
8. $3^{a+2b-7} \cdot 5^{3b-2a+4}$ by $3^{b+a-8} \cdot 5^{2b-2a+3}$.
9. $a^{3+2m-3n}b^5c^{7-n}$ by $a^{2+m-4n}b^4c^{7-n}$.
10. $x^{4a+1}y^{c-a}z^{3a+c}$ by $z^{-c+3a}y^{a-c}x^{3a-2c}$.
11. $2^{3a-4} \cdot 3^{3b+6}$ by $2^{2a-5} \cdot 3^{2b+7}$.
12. $(x-2)^{3m+1} \cdot (x+2)^{2m+2}$ by $(x+2)^{1+2m} \cdot (x-2)^{1+2m}$.
13. $(x-y)^{5b-1} \cdot (x+y)^{-2b+2}$ by $(x-y)^{-2+5b} \cdot (x+y)^{-3-2b}$.
14. $(a^2-b^2)^{3+4k} \cdot (a^2+b^2)^{1-5b}$ by $(a^2-b^2)^4 \cdot (a^2+b^2)^{-2+2b}$.

MULTIPLICATION OF POLYNOMIALS

48. Rule. *The product of two polynomials is equal to the sum of the products obtained by multiplying each term of one polynomial by every term of the other.*

This follows from the distributive law of multiplication, § 28. For by this law,

$$\begin{aligned}(m+n+k)(a+b+c) &= m(a+b+c)\\ &\quad + n(a+b+c)\\ &\quad + k(a+b+c).\end{aligned}$$

Applying the same law to each part, we have the product,

$$ma + mb + mc + na + nb + nc + ka + kb + kc.$$

WRITTEN EXERCISES

Find the following indicated products:

1. $(a+b)(a^2+3ab+2b^2)$.
2. $(x-y)(3x^3-2x^2y-4xy^2+2y^3)$.
3. $(x+y)(x^2-2xy+y^2)$.
4. $(x^2+5x+6)(x^2-3x+6)$.
5. $(3x^2-2x+6)(2x^2-3x+6)$.
6. $(2x+3y)(3x+2y)(x-y)$.
7. $(a^2+2ab+b^2)(a^2-2ab+b^2)$.
8. $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$.
9. $(x^2+y^2)(x^2-y^2)(x^4-y^4)$.
10. $(x^3+y^3)(x^3-y^3)(x^6+y^6)$.
11. $(16x^4+4x^2+1)(16x^4-4x^2+1)$.
12. $(x^3-4x^2+5x-1)(x^2+7x-3)$.
13. $(32x^5-16x^4+8x^3-4x^2+2x-1)(2x+1)$.
14. $(a^2b^2-3ab+6)(3+4ab-a^2b^2)$.
15. $(2x^2y^4-3x^3y+2xy^2)(xy^3-x^2y+x^3y^2)$.
16. $(x^3-y^3)(x^6+x^3y^3+y^6)$.
17. $(x^4+y^4)(x^8-x^4y^4+y^8)$.

49. Special Products. There are many special products which should be known at sight.

Example 1. $(a+b)(a-b)=a^2-b^2.$

The product of the sum and difference of two numbers equals the difference of their squares.

Example 2.
$$(a+b)^2=a^2+2ab+b^2,$$
$$(a-b)^2=a^2-2ab+b^2.$$

The square of the sum of two numbers equals the sum of their squares plus twice their product.

The square of the difference of two numbers equals the sum of their squares minus twice their product.

Example 3. $(x+a)(x+b)=x^2+(a+b)x+ab.$

The product of two binomials having a common term equals the square of the common term plus the sum of the unlike terms multiplied by the common term, plus the product of the unlike terms.

Example 4. $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$

The square of a polynomial equals the sum of the squares of the terms plus twice the product of each term taken with each term that succeeds it.

ORAL EXERCISES

1. $(a+b)(a-b)$.
2. $(x-2y)(x+2y)$.
3. $(3x+2y)(3x-2y)$.
4. $(x+y)^2$.
5. $(x-y)^2$.
6. $(a^2+b^2)^2$.
7. $(a^2-b^2)^2$.
8. $(2x+3y)^2$.
9. $(3x-5y)^2$.
10. $(a+b+c)^2$.
11. $(a+b-c)^2$.
12. $(a-b+c)^2$.
13. $(a+b)(a-b)(a^2+b^2)$.
14. $(x^4-y^4)(x^4+y^4)$.
15. $(x^2+x+1)(x^2+x-1)$.
16. $(x+1)(x+2)$.
17. $(x-2)(x-1)$.
18. $(x+3)(x-2)$.
19. $(x-3)(x+2)$.
20. $(x+a)(x+b)$.
21. $(x-a)(x+b)$.
22. $(x+a)(x-b)$.

DIVISION OF POLYNOMIALS

50. Dividing by a Monomial. According to the distributive law of division, § 29, a *polynomial is divided by a monomial* by dividing each term separately by the monomial.

A **polynomial is divided by a polynomial** by separating the dividend into parts, each of which is the product of the divisor and a monomial. Each of these monomial factors is a part of the quotient, their sum constituting the whole quotient. The parts of the dividend are found one by one as the work proceeds. This is best shown by an example.

Example 1.

$$\begin{array}{lr|ll}
\text{Dividend,} & a^4 + \quad a^3 - 4a^2 + 5a - 3 & a^2 + 2a - 3, & \text{Divisor.} \\
\text{1st part of dividend:} & \underline{a^4 + 2a^3 - 3a^2 \qquad\qquad} & a^2 - a + 1, & \text{Quotient.} \\
 & -a^3 - a^2 + 5a - 3 & & \\
\text{2d part of dividend:} & \underline{-a^3 - 2a^2 + 3a \quad\;} & & \\
 & a^2 + 2a - 3 & & \\
\text{3d part of dividend:} & \underline{a^2 + 2a - 3} & & \\
 & 0 & &
\end{array}$$

The three parts of the dividend are the products of the divisor and the three terms of the quotient. If after the successive subtraction of these parts of the dividend the remainder is zero, the division is exact. In case the division is not exact, there is a final remainder such that

$$\textit{Dividend} = \textit{Quotient} \times \textit{divisor} + \textit{Remainder}.$$

Example 2. Divide $x^3 - 7x^2 + 2x^4 + 6 + 5x$ by $2x - 1 + x^2$.

Solution. We first arrange both dividend and divisor according to the descending powers of x.

$$\begin{array}{lr|l}
 & & \text{[divisor.} \\
\text{Dividend or product} = & 2x^4 + \quad x^3 - 7x^2 + 5x + 6 & x^2 + 2x - 1, \\
\quad\text{1st product, } 2x^2(x^2 + 2x - 1) = & \underline{2x^4 + 4x^3 - 2x^2 \qquad\qquad} & 2x^2 - 3x + 1, \\
\text{Dividend minus 1st product} = & -3x^3 - 5x^2 + 5x + 6 & \text{[quotient.} \\
\quad\text{2d product, } -3x(x^2 + 2x - 1) = & \underline{-3x^3 - 6x^2 + 3x \quad\;} & \\
\text{Dividend minus 1st and 2d products} = & +x^2 + 2x + 6 & \\
\quad\text{3d product, } 1 \cdot (x^2 + 2x - 1) = & \underline{x^2 + 2x - 1} & \\
\text{Dividend minus 1st, 2d, and 3d products} = & 7 & = \text{remainder.}
\end{array}$$

51. From a consideration of the preceding examples the process of dividing by a polynomial is described as follows:

1. *Arrange the terms of dividend and divisor according to descending (or ascending) powers of some common letter. As the division proceeds, arrange each remainder in the same way.*

2. *Divide the first term of the dividend by the first term of the divisor. This result is the first term of the quotient.*

3. *Multiply the divisor by the first term of the quotient and subtract the product from the dividend.*

4. *Divide the first term of this remainder by the first term of the divisor, obtaining the second term of the quotient. Multiply the divisor by the second term of the quotient and subtract, obtaining a second remainder.*

5. *Continue in this manner until the last remainder is zero, or until a remainder is found whose first term does not contain as a factor the first term of the divisor.*

WRITTEN EXERCISES

Divide:

1. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.
2. $x^8 + x^4y^4 + y^8$ by $x^4 - x^2y^2 + y^4$.
3. $x^5 - y^5$ by $x - y$.
4. $x^8 - y^8$ by $x^3 + x^2y + xy^2 + y^3$.
5. $x^9 + y^9$ by $x^2 - xy + y^2$.
6. $x^8 - y^8$ by $x^2 - y^2$.
7. $a^6 + b^6$ by $a^2 + b^2$.
8. $x^{6a} - y^{6b}$ by $x^{2a} - y^{2b}$.
9. $a^{10} - a^5b^5 + b^{10}$ by $a^2 - ab + b^2$.
10. $2x^4 - 3x^3b + 6x^2b^2 - xb^3 + 6b^4$ by $x^2 - 2xb + 3b^2$.
11. $2x^6 - 5x^5 + 6x^4 - 6x^3 + 6x^2 - 4x + 1$ by $x^4 - x^3 + x^2 - x + 1$.
12. $26a^3b^3 + a^6 + 6b^6 - 5a^5b - 17ab^5 - 2a^4b^2 - a^2b^4$ by $a^2 - 3b^2 - 2ab$.
13. $x^4 + 2x^3 - 7x^2 - 8x + 12$ by $x^2 - 3x + 2$.
14. $4b^2 + 4ab + a^2 - 12bc - 6ac + 9c^2$ by $2b + a - 3c$.
15. $x^4 + 4xy^3 - 4xyz + 3y^4 + 2y^2z - z^2$ by $x^2 - 2xy + 3y^2 - z$.
16. $a^2b^2c + 3a^2b^3 - 3abc^3 - a^2c^3 + b^5 - 4b^3c^2 + 3ab^3c + 3bc^4 - 3a^2bc^2$ by $b^2 - c^2$.

52. Division by Detached Coefficients. The work of dividing by a polynomial may be shortened by omitting the letters and writing only the coefficients.

Example. Divide $2x^4 + x^3 - 7x^2 + 5x + 6$ by $x^2 + 2x - 1$.

Writing coefficients only,

$$\begin{array}{r|l}
2+1-7+5+6 & 1+2-1 \\ \cline{2-2}
\underline{2+4-2} & 2-3+1 \\
-3-5+5+6 & \\
\underline{-3-6+3} & \\
+1+2+6 & \\
\underline{+1+2-1} & \\
7 &
\end{array}$$

Since dividend and divisor are arranged in descending powers of x, the quotient will be so arranged also. Since the highest power of x in the quotient is x^2 the result is $2x^2 - 3x + 1$.

This example is worked without detached coefficients on page 22.

If any terms are lacking in the dividend or divisor, zero must be written as the coefficient of each such term, as in the following example.

Example. Divide $x^4 + x^2 + 1$ by $x^2 - x + 1$.

Since the third and first powers are lacking in the dividend, we write zero in place of each.

$$\begin{array}{r|l}
1+0+1+0+1 & 1-1+1 \\ \cline{2-2}
\underline{1-1+1} & 1+1+1 \\
+1+0+0+1 & \\
\underline{+1-1+1} & \\
+1-1+1 & \\
\underline{+1-1+1} &
\end{array}$$

Hence, the quotient starts with $x^4 \div x^2 = x^2$, and is $x^2 + x + 1$.

WRITTEN EXERCISES

Divide the following by detached coefficients:

1. $x^4 + x^3 - 12x^2 + 14x - 4$ by $x^2 - 3x + 2$.
2. $2x^4 + 11x^3 - 26x^2 + 16x - 3$ by $x^2 + 7x - 3$.
3. $x^5 - x^4 - 27x^3 + 10x^2 - 30x - 200$ by $x^2 - 4x - 10$.
4. $x^5 - 1$ by $x - 1$.

53. Synthetic Division. When the divisor is of the form $x - a$, the work may be contracted still further.

Example 1. Divide $x^4 - 3x^3 + 3x^2 - 2x + 1$ by $x - 1$.

FIRST SOLUTION

$$\begin{array}{r|l}
1-3+3-2+1 & 1-1 \\ \cline{2-2}
\underline{1-1} & 1-2+1-1 \\
-2+3 & \\
\underline{-2+2} & \\
1-2 & \\
\underline{1-1} & \\
-1+1 & \\
\underline{-1+1} & \\
0 &
\end{array}$$

SECOND SOLUTION

$$\begin{array}{r|l}
1-3+3-2+1 & 1-1 \\ \cline{2-2}
\underline{-1} & 1-2+1-1 \\
-2+3 & \\
\underline{+2} & \\
1-2 & \\
\underline{-1} & \\
-1+1 & \\
\underline{1} & \\
0 &
\end{array}$$

In the second solution the first term in each partial product is omitted.

In practice the first term of the divisor is also omitted, and the figures below the dividend are moved up as shown below. The sign of the second term of the divisor is changed, and the partial products are *added* instead of *subtracted*.

$$\begin{array}{r|l}
1-3+3-2+1 & +1 \\ \cline{2-2}
\underline{+1-2+1-1} & \\
-2+1-1+0 &
\end{array}$$

Multiply the first coefficient of the dividend by $+1$, and add to the second coefficient. Multiply the sum, -2, by $+1$, and add to the third coefficient. Multiply the sum, $+1$, by $+1$, and add to the fourth coefficient. Multiply the sum, -1, by $+1$, and add to the last coefficient.

The coefficient of the highest power in the quotient is the same as that of the highest power of the dividend. The remaining coefficients of the quotient are the sums below the line in the solution. The last sum to the right is the remainder.

This method of dividing is called synthetic division.

Example 2. Divide $x^5 - 3x^4 - 4x^2 - 7x + 9$ by $x - 2$.

$$\begin{array}{r|l}
1-3+0-4-7+9 & +2 \\ \cline{2-2}
\underline{+2-2-4-16-46} & \\
-1-2-8-23-37 &
\end{array}$$

The quotient is $x^4 - x^3 - 2x^2 - 8x - 23$, and the remainder is -37.

WRITTEN EXERCISES

Divide the following, using synthetic division.

1. $x^3 + 4x^2 + x - 6$ by $x - 1$.
2. $x^3 - 8x^2 + 75$ by $x - 5$.
3. $x^4 - 3x^2 + 4x - 66$ by $x - 3$.
4. $x^7 - 1$ by $x - 1$.
5. $x^6 - 1$ by $x + 1$.
6. $x^4 + x^3 - 12x^2 + 14x - 4$ by $x - 2$.
7. $x^4 - 6x^3 + 2x^2 - 3x + 6$ by $x - 1$.
8. $x^4 - 3x^3 + 7x^2 - 4x + 3$ by $x - 4$.

54. Special Quotients. There are many special quotients which should be known at sight. The test in every case is

Quotient $\times$ *Divisor* = *Dividend.*

Example 1. $(a^2 - b^2) \div (a - b) = a + b$, and $(a^2 - b^2) \div (a + b) = a - b$.

Example 2. $(a^2 + 2ab + b^2) \div (a + b) = a + b$, and $(a^2 - 2ab + b^2) \div (a - b) = a - b$.

Example 3. $(x^2 + 5x + 6) \div (x + 2) = x + 3$, and $(x^2 + 5x + 6) \div (x + 3) = x + 2$.

ORAL EXERCISES

1. $(a^2 - b^2) \div (a + b)$.
2. $(a^2 - b^2) \div (a - b)$.
3. $(a^2 + 2ab + b^2) \div (a + b)$.
4. $(a^2 - 2ab + b^2) \div (a - b)$.
5. $(x^4 - y^4) \div (x^2 - y^2)$.
6. $(x^4 - y^4) \div (x^2 + y^2)$.
7. $(x^6 + y^6) \div (x^3 + y^3)$.
8. $(x^6 - y^6) \div (x^3 - y^3)$.
9. $(x^2 + 3x + 2) \div (x + 1)$.
10. $(x^2 + 3x + 2) \div (x + 2)$.
11. $(x^2 - 3x + 2) \div (x - 1)$.
12. $(x^2 - 3x + 2)(\div x - 2)$.
13. $(x^2 + x - 2) \div (x + 2)$.
14. $(x^2 - x - 2) \div (x - 2)$.
15. $(4x^2 - 9y^2) \div (2x + 3y)$.
16. $(x^2 + 8x + 16) \div (x + 4)$.
17. $(x^2 + 7x + 12) \div (x + 4)$.
18. $(625 - 16a^2) \div (25 - 4a)$.

CHAPTER III

FACTORING

55. Rational Integral Expressions. An algebraic expression which is rational in a certain letter (see § 15), and in which no denominator contains the letter, is said to be *rational and integral in that letter.*

56. Prime Factors. A rational and integral expression is said to be *completely factored* when it cannot be further resolved into factors which are rational and integral. Such factors are called *prime factors.*

MONOMIAL FACTORS

57. A monomial factor of an expression is evident at sight and its removal should be the first step in every case.

That is, $\qquad ad + bd + cd = (a + b + c)d.$

E.g. $\qquad 4\,ax^2 + 2\,a^2x = 2\,ax(2\,x + a).$

ORAL EXERCISES

Give the factors of each of the following:

1. $a^3 + a^2 + a.$
2. $2\,x^3 + 3\,x^2 + x.$
3. $a^2b + ab^2.$
4. $3\,a^4 + 6\,a^2 + 9\,a.$
5. $4\,x^3 + 8\,x^4 - 12\,x^5.$
6. $7\,x^2y^3 + 21\,x^4y^2.$
7. $2\,a^3b^2 + 4\,a^2b^3.$
8. $ab + ac + ad.$
9. $3\,c + 6\,b + 9\,a.$
10. $4\,a + 8\,b + 12\,c.$
11. $3\,a^2 + 9\,ab + 4\,a.$
12. $4\,x^2y^2 + 12\,x^4y.$
13. $6\,a^4x^2 - 12\,a^5x^3 + 18\,x^5.$
14. $7\,ab^3 + 14\,a^2b^2 + 21\,a^3b^3.$
15. $5\,x^3y^2 - 10\,x^2y^2 + 15\,x^2y^3.$
16. $4\,a^4x^2 - 6\,a^5x^2 + 8\,a^2x^3.$
17. $10\,x^7 - 15\,x^8 + 20\,x^9.$
18. $13\,axy + 26\,a^2x^2y^2.$

FACTORS OF BINOMIALS

58. The Difference of Two Squares.

$$a^2 - b^2 = (a - b)(a + b).$$

E.g. $4x^2 - 9z^4 = (2x + 3z^2)(2x - 3z^2).$

ORAL EXERCISES

Factor the following:

1. $a^2 - b^2.$	7. $4x^2 - 9y^2.$	13. $x^2 - y^6.$
2. $4x^2 - y^2.$	8. $9x^2 - 4y^2.$	14. $x^6 - y^2.$
3. $9x^2 - y^2.$	9. $x^4 - y^4.$	15. $x^4 - y^6.$
4. $x^2 - 4y^2.$	10. $x^2 - y^4.$	16. $x^6 - y^4.$
5. $x^2 - 9y^2.$	11. $x^4 - y^2.$	17. $25 - 4x^2.$
6. $4x^2 - 4y^2.$	12. $x^6 - y^6.$	18. $25a^2 - 4b^2$

59. The Difference of Two Cubes.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

E.g.
$$\begin{aligned} 8x^3 - 27y^3 &= (2x - 3y)[(2x)^2 + (2x)(3y) + (3y)^2] \\ &= (2x - 3y)(4x^2 + 6xy + 9y^2). \end{aligned}$$

ORAL EXERCISES

Factor the following:

1. $a^3 - b^3.$	4. $a^3 - 8.$	7. $a^6 - 1.$
2. $1 - b^3.$	5. $8 - b^3.$	8. $1 - a^6.$
3. $a^3 - 1.$	6. $a^6 - b^3.$	9. $a^3 - b^6.$

60. The Sum of Two Cubes.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

E.g.
$$\begin{aligned} 27x^3 + 64y^3 &= (3x + 4y)[(3x)^2 - (3x)(4y) + (4y)^2] \\ &= (3x + 4y)(9x^2 - 12xy + 16y^2). \end{aligned}$$

ORAL EXERCISES

Factor the following:

1. $a^3 + b^3.$	4. $x^3 + b^6.$	7. $a^3 + 8.$
2. $a^3 + 1.$	5 $a^6 + b^6.$	8. $a^6 + 8.$
3. $1 + b^3.$	6. $8 + b^3.$	9. $8 + a^6.$

FACTORS OF TRINOMIALS

61. Trinomial Squares.

$$a^2 + 2ab + b^2 = (a + b)^2 = (a + b)(a + b),$$
$$a^2 - 2ab + b^2 = (a - b)^2 = (a - b)(a - b).$$

ORAL EXERCISES

Factor the following:

1. $x^2 + 2xy + y^2$.
2. $x^2 - 2xy + y^2$.
3. $x^2 + 4x + 4$.
4. $x^2 - 4x + 4$.
5. $x^2 + 4xy + 4y^2$.
6. $x^2 - 4xy + 4y^2$.
7. $4x^2 + 4x + 1$.
8. $4x^2 - 4x + 1$.
9. $4x^2 - 4xy + y^2$.
10. $4x^2 + 4xy + y^2$.
11. $4a^2 + 12ab + 9b^2$.
12. $4a^2 - 12ab + 9b^2$.
13. $a^2 + 6ab + 9b^2$.
14. $x^2 + 8xy + 16y^2$.
15. $9x^2 + 24xy + 16y^2$.
16. $9x^2 - 24xy + 16y^2$.

62. Trinomials of the Form $x^2 + px + q$.

$$x^2 + (a + b)x + ab = (x + a)(x + b).$$

E.g. $\qquad x^2 + 3x - 10 = (x + 5)(x - 2).$

A trinomial of this form has two binomial factors, $x + a$ and $x + b$, if two numbers a and b can be found whose product is q, and whose algebraic sum is p.

ORAL EXERCISES

Factor the following:

1. $x^2 + 3x + 2$.
2. $x^2 - 3x + 2$.
3. $x^2 + 5x + 6$.
4. $x^2 - x - 2$.
5. $x^2 + x - 2$.
6. $x^2 - x - 6$.
7. $x^2 + 2x - 15$.
8. $x^2 + x - 6$.
9. $x^2 - 2x - 15$.
10. $x^2 + 8x + 15$.
11. $x^2 - 8x + 15$.
12. $x^2 - 5x - 6$.
13. $x^2 + 5x - 6$.
14. $x^2 + 7x + 12$.
15. $x^2 - 9x + 20$.
16. $x^2 - 5x - 24$.
17. $x^2 + 6x - 16$.
18. $x^2 - 7x - 30$.

63. Trinomials of the Form $mx^2 + nx + r$.

E.g. $6x^2 + 7x - 20 = (3x - 4)(2x + 5)$.

A trinomial of this form has two binomial factors of the type $ax + b$ and $cx + d$, if four numbers, a, b, c, d, can be found, such that $ac = m$, $bd = r$, and $ad + bc = n$.

64. Trinomials which Reduce to the Difference of Two Squares.

E.g. $$x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2 = (x^2 + y^2)^2 - x^2y^2.$$
$$= (x^2 + y^2 - xy)(x^2 + y^2 + xy).$$

In this case x^2y^2 is both added to and subtracted from the expression, whereby it becomes the difference of two squares.

Example.

$$4a^8 - 16a^4b^4 + 9b^8 = 4a^8 - 12a^4b^4 + 9b^8 - 4a^4b^4.$$
$$= (2a^4 - 3b^4)^2 - 4a^4b^4.$$
$$= (2a^4 - 3b^4 + 2a^2b^2)(2a^4 - 3b^4 - 2a^2b^2).$$

WRITTEN EXERCISES

Factor the following:

1. $a^3 + b^3$.
2. $a^3 - b^3$.
3. $(a + b)^3 + c^3$.
4. $(a + b)^3 - c^3$.
5. $7ax^2 - 56a^4x^5$.
6. $a^5 - ab^4$.
7. $121x^7 - 4xy^4$.
8. $\frac{1}{8}a^3 + \frac{1}{125}b^3$.
9. $\frac{1}{8}r^3 - \frac{9}{32}rs^2$.
10. $8r^4 - 27r$.
11. $(a + b)^2 - c^2$.
12. $c^2 - (a - b)^2$.
13. $5c^2 + 7cd - 6d^2$.
14. $x^4 - 3x^2y^2 + y^4$.
15. $4x^2 - 12xy + 9y^2$.
16. $x^2 + 11xz + 30z^2$.
17. $6x^2 - 5xy - 6y^2$.
18. $3a^2x^2y^4 - 69a^2xy^2 + 336a^2$.
19. $20a^2b^2 + 23abx - 21x^2$.
20. $a^4 + 2a^2b^2 + 9b^4$.
21. $48a^3x^4y - 75ay^5$.
22. $16a^4x^3y + 54ay^4$.
23. $x^4y^2 + 2x^2yz + z^2$.
24. $a^2 + 10a - 39$.
25. $8a^2y^3 - 48a^2y^2z + 72a^2yz^2$.
26. $4m^8 - 60m^4n^4 + 81n^8$.
27. $35a^2 - 6ab - 9b^2$.
28. $(a + b)^2 - (c - d)^2$.
29. $72a^2x^2 - 19axy^2 - 40y^4$.
30. $4(a-3)^6 - 37b^2(a-3)^3 + 9b^4$.

FACTORS OF POLYNOMIALS OF FOUR TERMS

A polynomial of four terms may be readily factored if it is in one of the forms given in the next three paragraphs:

65. *It may be the cube of a binomial.*

Example 1. $a^3 - 3\,a^2b + 3\,ab^2 - b^3 = (a - b)^3$.

Example 2. $8\,x^3 + 36\,x^2y + 54\,xy^2 + 27\,y^3$

$= (2\,x)^3 + 3(2\,x)^2(3\,y) + 3(2\,x)(3\,y)^2 + (3\,y)^3 = (2\,x + 3\,y)^3$.

66. *It may be resolvable into the difference of two squares.*

In this case three of the terms must form a trinomial square.

Example 1. $a^2 - c^2 + 2\,ab + b^2 = (a^2 + 2\,ab + b^2) - c^2$

$= (a + b)^2 - c^2 = (a + b + c)(a + b - c)$.

Example 2. $4\,x^2 + z^6 - 4\,x^4 - 1 = z^6 - (4\,x^4 - 4\,x^2 + 1)$

$= z^6 - (2\,x^2 - 1)^2 = (z^3 + 2\,x^2 - 1)(z^3 - 2\,x^2 + 1)$.

67. *A binomial factor may be found by grouping the terms.*

In this case the terms are grouped by twos as in the following examples.

Example 1. $ax + ay + bx^2 + bxy = (ax + ay) + (bx^2 + bxy)$

$= a(x + y) + bx(x + y) = (a + bx)(x + y)$.

Example 2. $ax + bx + a^2 - b^2 = (ax + bx) + (a^2 - b^2)$

$= x(a + b) + (a - b)(a + b) = (x + a - b)(a + b)$.

WRITTEN EXERCISES

Factor the following polynomials:

1. $x^3 + 3\,x^2y + 3\,xy^2 + y^3$.
2. $8\,a^3 - 36\,a^2b + 54\,ab^2 - 27\,b^3$.
3. $4\,a^4 - 4\,a^2b^2 + b^4 - 16\,x^2$.
4. $2\,ad + 3\,bc + 2\,ac + 3\,bd$.
5. $27\,x^3 - 54\,x^2y + 36\,xy^2 - 8\,y^3$.
6. $36\,a^4 - 24\,a^3 + 24\,a - 16$.
7. $mnx^2 - mrx - rn^2x + r^2n$.
8. $a^2b^2 - a^2bc^2n - abn + an^2c^2$.
9. $2\,y^2 + 4\,by + 3\,cy + 6\,bc$.
10. $bcyz + c^2z^2 + bdy + cdz$.
11. $5\,a^2c + 12\,cd - 6\,ad - 10\,ac^2$.
12. $a^2 - b^2x^2 + acx^2 - bcx^3$.
13. $b^3c^2 - c^2y^3 - b^3y^2 + y^5$.
14. $m^{a+b} + m^an^a + m^bn^b + n^{a+b}$.

FACTORS FOUND BY GROUPING

68. The discovery of factors by the proper grouping of terms is of wide application. Polynomials of five, six, or more terms may frequently be thus resolved into factors.

Example 1. $a^2 + 2ab + b^2 + 5a + 5b = (a+b)^2 + (5a + 5b)$
$= (a+b)(a+b) + 5(a+b) = (a+b+5)(a+b).$

Example 2. $x^2 - 7x + 6 - ax + 6a = x^2 - 7x + 6 - (ax - 6a)$
$= (x-1)(x-6) - a(x-6) = (x-6)(x-1-a).$

Example 3. $a^2 - 2ab + b^2 - x^2 + 2xy - y^2$
$= (a^2 - 2ab + b^2) - (x^2 - 2xy + y^2).$
$= (a-b)^2 - (x-y)^2 = (a-b+x-y)(a-b-x+y)$

Example 4. $ax^2 + ax - 6a + x^2 + 7x + 12$
$= a(x^2 + x - 6) + (x^2 + 7x + 12)$
$= a(x+3)(x-2) + (x+3)(x+4)$
$= (x+3)[a(x-2) + x + 4] = (x+3)(ax - 2a + x + 4).$

In some cases the grouping is effected only after a term has been separated into two parts.

Example 5. $2a^3 + 3a^2 + 3a + 1 = a^3 + (a^3 + 3a^2 + 3a + 1)$
$= a^3 + (a+1)^3 = (a + a + 1)[a^2 - a(a+1) + (a+1)^2]$
$= (2a+1)(a^2 + a + 1).$

As soon as the term $2a^3$ is separated into two terms the expression is shown to be the sum of two cubes.

Again, the grouping may be effective after a term has been both added and subtracted:

Example 6. $x^8 + x^4y^4 + y^8 = x^8 + 2x^4y^4 + y^8 - x^4y^4$
$= (x^4 + y^4 - x^2y^2)(x^4 + y^4 + x^2y^2)$
$= (x^4 + y^4 - x^2y^2)(x^2 + y^2 - xy)(x^2 + y^2 + xy).$

Example 7. $a^4 + b^4 = (a^4 + 2a^2b^2 + b^4) - 2a^2b^2$
$= (a^2 + b^2)^2 - (ab\sqrt{2})^2$
$= (a^2 + b^2 + ab\sqrt{2})(a^2 + b^2 - ab\sqrt{2}).$

In this case the factors are irrational as to one coefficient. Such factors are often useful in higher mathematical work.

WRITTEN EXERCISES

Factor the following:

1. $x^2 - 2xy + y^2 - ax + ay.$
2. $a^2 - ab + b^2 + a^3 + b^3.$
3. $a^3 - b^3 - a^2 - ab - b^2.$
4. $a^2 - 2ab + b^2 - x^2 + 2xy - y^2.$
5. $a^4 + 2a^3b - a^2c^2 + a^2b^2 - 2abc^2 - b^2c^2.$
6. $x^4 - y^4 + ax^2 + ay^2 - x^2 - y^2.$
7. $a^4 + a^2b^2 + b^4 + a^3 + b^3.$

In example 7 group the first three and the last two terms and then add and subtract a^2b^2.

8. $a^3 - 1 + 3x - 3x^2 + x^3.$ Group the last four terms.
9. $x^3 + x^2 + 3x + y^3 - y^2 + 3y.$

Group in pairs, the 1st and 4th, 2d and 5th, 3d and 6th terms.

10. $x^4 + x^3y - xy^3 - y^4 + x^3 - y^3.$
11. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 - x^4.$
12. $x^4 + 4x^2z - 4y^2 + 4yw + 4z^2 - w^2.$
13. $2a^2 - 12b^2 + 3bd - 5ab - 9bc - 6ac + 2ad.$

Group the terms: $2a^2 - 5ab - 12b^2$.

14. $a^2 + ab - 4ac - 2b^2 + 4bc + 3ad - 3bd.$
15. $a^3 + 2 - 3a.$

Add and subtract 1 and group thus: $(a^3 - 1) + (3 - 3a)$.

16. $4a^2 + a - 8ax - x + 4x^2.$
17. $3a^2 - 8ab + 4b^2 + 2ac - 4bc.$
18. $a^6 + 2a^3b^3 + b^6 - 2a^4b - 2ab^4.$
19. $a^3 - 3a^2 + 4.$

Group thus: $(a^3 - 2a^2) + (4 - a^2)$.

20. $a^2c - ac^2 - a^2b + ab^2 - b^2c + bc^2.$
21. $a^2b - a^2c + b^2c - ab^2 + ac^2 - bc^2.$
22. $3x^3 - x^2 - 4x + 2.$ Add and subtract $-2x^2$.
23. $2x^3 - 11x^2 + 18x - 9.$ Add and subtract $9x^2$.
24. $x^8 + y^8.$ Add and subtract $2x^4y^4$.

FACTORS FOUND BY THE REMAINDER THEOREM

69. It is possible to determine in advance whether a polynomial in x is divisible by a binomial of the form $x - a$.

E.g. In dividing $x^4 - 4x^3 + 7x^2 - 7x + 2$ by $x - 2$, the quotient is found to be $x^3 - 2x^2 + 3x - 1$.

Since *Quotient* × *Divisor* = *Dividend*, we have

$$(x-2)(x^3 - 2x^2 + 3x - 1) = x^4 - 4x^3 + 7x^2 - 7x + 2.$$

As this is an *identity*, it holds for all values of x. For $x = 2$ the factor $(x - 2)$ is zero, and hence the left member is zero.

Hence for $x = 2$ the right member must also be zero. This is indeed the case, viz.:

$$2^4 - 4 \cdot 2^3 + 7 \cdot 2^2 - 7 \cdot 2 + 2 = 16 - 32 + 28 - 14 + 2 = 0.$$

Hence, if $x - 2$ is a factor of $x^4 - 4x^3 + 7x^2 - 7x + 2$, the latter must reduce to zero for $x = 2$.

70. The Remainder Theorem. In general let D represent any polynomial in x. Suppose D has been divided by $x - a$ until the remainder no longer contains x. Then, calling the quotient Q and the remainder R, we have the identity

$$D = Q(x - a) + R, \qquad (1)$$

which holds for all values of x.

The substitution of a for x in (1) does not affect R, reduces $Q(x - a)$ to zero, and may or may not reduce D to zero.

(1) If $x = a$ reduces D to zero, then $0 \equiv 0 + R$. Hence R is zero, and the division is exact. That is, $x - a$ is a factor of D.

(2) If $x = a$ does *not* reduce D to zero, then R is not zero, and the division is not exact. That is, $x - a$ is not a factor of D.

Hence: *If a polynomial in x reduces to zero when a particular number a is substituted for x, then $x - a$ is a factor of the polynomial, and if the substitution of a for x does not reduce the polynomial to zero, then $x - a$ is not a factor.*

This principle is called the **Remainder theorem.**

In applying the remainder theorem the trial divisor must always be written in the form $x - a$.

Example 1. Factor $x^4 + 6x^3 + 3x^2 + x + 3$.

If there is a factor of the form $x - a$, then the only possible values of a are the various divisors of 3, namely $+1$, -1, $+3$, -3.

To test the factor $x + 1$, we write it in the form $x - (-1)$ where $a = -1$. Substituting -1 for x in the polynomial, we have

$$1 - 6 + 3 - 1 + 3 = 0.$$

Hence, $x + 1$ is a factor.

On substituting $+1$, $+3$, -3 for x successively, no one reduces the polynomial to zero. Hence, $x - 1$, $x - 3$, $x + 3$ are not factors.

Example 2. Factor $3x^3 - x^2 - 4x + 2$.

If $x - a$ is a factor, then a must be a factor of $+2$. We therefore substitute, $+2$, -2, $+1$, -1 and find the expression becomes zero when $+1$ is substituted for x. Hence, $x - 1$ is a factor. The other factor is found by division to be $3x^2 + 2x - 2$, which is prime.

Hence,

$$3x^3 - x^2 - 4x + 2 = (x - 1)(3x^2 + 2x - 2).$$

ORAL EXERCISES

1. Is $x - 1$ a factor of $x^3 - 4x^2 + 5x - 2$?
2. Is $x - 1$ a factor of $x^3 + 2x^2 - 7x + 3$?
3. Is $x - 2$ a factor of $x^3 + x^2 - x - 10$?
4. Is $x - 2$ a factor of $x^3 - 2x^2 + 4x - 8$?
5. Is $x + 2$ a factor of $x^3 - 2x^2 + 4x - 8$?
6. Is $x - 3$ a factor of $x^3 - 3x^2 - 4x + 12$?
7. Is $x - 3$ a factor of $x^3 + 3x^2 - 15x - 9$?
8. Is $x + 2$ a factor of $x^3 + 3x^2 - 2x - 10$?
9. Is $x + 3$ a factor of $x^3 + 4x^2 + 2x - 3$?
10. Is $x + 3$ a factor of $x^3 - 4x^2 - 2x - 6$?
11. Is $x + 1$ a factor of $x^4 + 3x^3 - 4x^2 + x + 7$?
12. Is $x + 1$ a factor of $x^5 - 2x^3 + 4x + 2$?
13. Is $x + 2$ a factor of $x^5 + 4x^2 + 16$?
14. Is $x + 2$ a factor of $x^5 + 5x^2 + 10$?

WRITTEN EXERCISES

Factor by means of the remainder theorem:

1. $3x^3 - 2x^2 + 5x - 6$.
2. $2x^3 + 3x^2 - 3x - 4$.
3. $2x^3 + x^2 - 12x + 9$.
4. $x^3 + 9x^2 + 10x + 2$.
5. $a^3 - 3a + 2$.
6. $m^3 + 5m^2 + 7m + 3$.
7. $x^4 + 3x^3 - 3x^2 - 7x + 6$.
8. $3r^3 + 5r^2 - 7r - 1$.
9. $2z^3 + 7z^2 + 4z + 3$.
10. $a^3 - 6a^2 + 11a - 6$.

71. Applying the Remainder Theorem. By use of the remainder theorem we may find under what conditions $x + y$ and $x - y$ are factors of $x^n + y^n$ and $x^n - y^n$.

1. Is $x + y$ a factor of $x^n + y^n$?

If we substitute $-y$ for x we have $(-y)^n + y^n$. This is zero only when $(-y)^n = -y^n$; that is, when n is an *odd integer*.

Hence $x + y$ is a factor of $x^5 + y^5$, but *not* of $x^6 + y^6$.

2. Is $x + y$ a factor of $x^n - y^n$?

Here we have $(-y)^n - y^n$. This reduces to zero only when $(-y)^n = +y^n$; that is, when n is an *even integer*.

Hence $x + y$ is a factor of $x^6 - y^6$, but *not* of $x^7 - y^7$.

3. Is $x - y$ a factor of $x^n + y^n$?

Since $y^n + y^n$ is never zero, $x - y$ is not a factor of $x^n + y^n$.
E.g. $x - y$ is not a factor of $x^3 + y^3$, nor of $x^4 + y^4$.

4. Is $x - y$ a factor of $x^n - y^n$?

Since $y^n - y^n = 0$, $x - y$ is a factor of $x^n - y^n$, for all integral values of n.
E.g. $x - y$ is a factor of $x^3 - y^3$ and also of $x^4 - y^4$.

Summary. From the foregoing examples, we conclude that:

(1) *$x + y$ is a factor of $x^n - y^n$ if n is **even** but not if n is **odd**.*
(2) *$x - y$ is a factor of $x^n - y^n$ whether n is **even** or **odd**.*
(3) *$x + y$ is a factor of $x^n + y^n$ if n is **odd** but not if n is **even**.*
(4) *$x - y$ is not a factor of $x^n + y^n$ in **any** case.*

MISCELLANEOUS EXERCISES IN FACTORING

1. $20a^3x^3y - 45a^3xy^3$.
2. $24am^5n^2 - 375am^2n^5$.
3. $432ar^4s + 54ars^4$.
4. $16x^2 - 72xy + 81y^2$.
5. $162a^3b + 252a^2b^2 + 98ab^3$.
6. $48x^5y - 12x^3y - 12x^2y + 3y$.
7. $12a^2bx^2 + 8ab^2x^2 + 18a^2bxy + 12ab^2xy$.
8. $18x^3y - 39x^2y^2 + 18xy^3$.
9. $4x^2 - 9xy + 6x - 9y + 4x + 6$.
10. $6x^2 - 13xy + 6y^2 - 3x + 2y$.
11. $6x^4 - 15x^2y^2 + 9y^4$.
12. $16x^4 + 24x^2y^2 + 8y^4$.
13. $15x^4 + 24x^2y^2 + 9y^4$.
14. $a^6 + y^6$.
15. $a^{12} + y^{12}$.
16. $a^8 - y^8$.
17. $a^{16} - y^{16}$.
18. $a^8 + a^4y^4 + y^8$.
19. $a^3 + a - 2$.
20. $a^8 - 18a^4y^4 + y^8$.

    Group as follows:
    $a^8 - 2a^4y^4 + y^8 - 16a^4y^4$.
21. $a^{16} - 6a^8y^8 + y^{16}$.
22. $x^3 + 4x^2 + 2x - 1$.
23. $3x^3 + 2x^2 - 7x + 2$.
24. $a^8 - 3a^4y^4 + y^8$.
25. $a^3 + a^2 + a + 1$.
26. $a^3 + 9a^2 + 16a + 4$.
27. $2x^4 + x^3y + 2x^2y^2 + xy^3$.
28. $(x-2)^3 - (y-z)^3$.
29. $a^6 + b^6 + 2ab(a^4 - a^2b^2 + b^4)$.
30. $8a^3 + 6ab(2a - 3b) - 27b^3$.
31. $a(x^3 + y^3) - ax(x^2 - y^2) - y^2(x + y)$.
32. $a^3 - b^3 + 3b^2c - 3bc^2 + c^3$.
33. $a^4 + 2a^3b - 2ab^2c - b^2c^2$.
34. $a^4 + 2a^3b + a^2b^2 - a^4b^2 - 2a^2b^2c - b^2c^2$.
35. $6(x+y)^2 + 5(x^2 - y^2) - 6(x-y)^2$.
36. $9(x-a)^2 - 24(x-a)(x+a) + 16(x+a)^2$.
37. $12(c+d)^2 - 7(c+d)(c-d) - 12(c-d)^2$.
38. $(a^2 + 5a - 3)^2 - 25(a^2 + 5a - 3) + 150$.
39. $2x^3 + x^2 - 6x + 3$.
40. $12x^4 + 14x^3 - 3x^2 - 4x + 1$.
41. $6x^3 - x^2 - 20x + 12$.
42. $x^5 - 4x^4 - 40x^3 - 58x^2 - x + 6$.

COMMON FACTORS AND MULTIPLES

72. Highest Common Factor. If each of two or more expressions is resolved into prime factors, then their *Highest Common Factor* (H. C. F.) is at once evident, as in the following example.

Given (1) $x^4 - y^4 = (x^2 + y^2)(x + y)(x - y)$,

(2) $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$

$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y)^2$.

Then $(x + y)(x - y) = x^2 - y^2$ is the H. C. F. of (1) and (2).

In case only one of the given expressions can be factored by inspection, it is usually possible to select those of its factors, if any, which will divide the other expressions and so to determine the H.C.F.

Example. Find the H.C.F. of $6x^3 + 4x^2 - 3x - 2$,
and $2x^4 + 2x^3 + x^2 - x - 1$.

By grouping we find:

$$6x^3 + 4x^2 - 3x - 2 = 2x^2(3x + 2) - (3x + 2)$$
$$= (2x^2 - 1)(3x + 2).$$

The other expression cannot readily be factored by any of the methods thus far studied. However, if there is a common factor, it must be either $2x^2 - 1$ or $3x + 2$. We see at once that it cannot be $3x + 2$. (Why?) By actual division $2x^2 - 1$ is found to be a factor of $2x^4 + 2x^3 + x^2 - x - 1$. Hence, $2x^2 - 1$ is the H.C.F.

ORAL EXERCISES

Find the H.C.F. of each of the following sets of expressions:

1. $x^2 + 2xy + y^2$, $x^2 - y^2$.
2. $x^2 - 2xy + y^2$, $x^2 - y^2$.
3. $x^2 + 3x + 2$, $x^2 - 4$.
4. $x^2 + 4x + 3$, $x^2 + 5x + 6$
5. $x^2 - 4y^2$, $x^2 + 4xy + 4y^2$.
6. $4x^2 - y^2$, $4x^2 - 4xy + y^2$.
7. $x^2 + 5x + 6$, $x^2 + 7x + 12$.
8. $x^2 - 2x - 3$, $x^2 - 6x + 9$.
9. $x^2 - 9y^2$, $x^2 + 6xy + 9y^2$.
10. $x^2 - 5x - 6$, $x^2 - 36$.
11. $x^2 + 3x - 10$, $x^2 - 4$.
12. $x^2 - 7x - 18$, $x^2 + 5x + 6$.
13. $x^4 - y^4$, $x^2 + y^2$.
14. $x^4 - y^4$, $x^2 - y^2$.
15. $x^2 + y^2$, $x^6 + y^6$.
16. $x^3 - y^3$, $x^2 + xy + y^2$.
17. $x^3 + y^3$, $x^2 - xy + y^2$.
18. $x^8 - y^8$, $x^4 - y^4$, $x^2 - y^2$.

73. Lowest Common Multiple. *The Lowest Common Multiple* (L. C. M.) of two or more expressions is readily found if the expressions are resolved into prime factors.

Example 1. Given $6\,abx - 6\,aby = 2 \cdot 3\,ab(x-y),$ (1)

$$8\,a^2x + 8\,a^2y = 2^3a^2(x+y), \quad (2)$$

$$36\,b^3(x^2-y^2)(x+y) = 2^23^2b^3(x-y)(x+y)^2. \quad (3)$$

The L. C. M. is $2^3 \cdot 3^2a^2b^3(x-y)(x+y)^2$, since this contains all the factors of (1), all the factors of (2) not found in (1), and all the factors of (3) not found in (1) and (2), and since there is no factor to spare.

In case only one of the given expressions can be factored by inspection, it may be found by actual division whether or not any of these factors will divide the other expressions.

Example 2. Find the L. C. M. of $6\,x^3 - x^2 + 4\,x + 3,$ (1)

and $6\,x^3 + 3\,x^2 - 10\,x - 5.$ (2)

(1) is not readily factored. Grouping by twos, the factors of (2) are $3\,x^2 - 5$ and $2\,x + 1$. Now $3\,x^2 - 5$ is not a factor of (1). (Why?) Dividing (1) by $2\,x + 1$ the quotient is $3\,x^2 - 2\,x + 3$.

Hence, $6\,x^3 - x^2 + 4\,x + 3 = (2\,x+1)(3\,x^2 - 2\,x + 3),$

$6\,x^3 + 3\,x^2 - 10\,x - 5 = (2\,x+1)(3\,x^2 - 5).$

Hence, the L. C. M. is $(2\,x+1)(3\,x^2 - 2\,x + 3)(3\,x^2 - 5)$.

Example 3. Find the L. C. M. of $a^3 + 2\,a^2 - a - 2,$ (1)

and $10\,a^3 - 3\,a^2 + 4\,a + 1.$ (2)

Using the remainder theorem, $a - 1$, $a + 1$, and $a + 2$ are found to be factors of (1), but none of the numbers, 1, -1, -2, when substituted for a in (2) will reduce it to zero. Hence, (1) and (2) have no factors in common. The L. C. M. is therefore the product of the two expressions; viz. $(a+1)(a-1)(a+2)(10\,a^3 - 3\,a^2 + 4\,a + 1)$.

Rule for finding Lowest Common Multiple. *To obtain the lowest common multiple of a set of expressions:*

(1) *Find the prime factors of each expression.*

(2) *Use all factors of the first expression, together with those factors of the next expression which are not in the first, those of the third which are neither in the first nor in the second, etc.*

ORAL EXERCISES

Find the L. C. M. of each the following sets of expressions. Give each result in the factored form.

1. $(x-2)(x-3), \quad (x-4)(x-3).$
2. $4(a-b)(b-c), \quad 2(b-c)(d-a).$
3. $3\,a^2(x+2)(x-1), \quad 6\,a^3(x-1)(x-2).$
4. $5(a+2\,b)(a-2\,b), \quad 3(a+2\,b)(a-4\,b).$
5. $12(x^2-y^2), \quad x-y, \; x+y.$
6. $3(x^2-4\,xy+4\,y^2), \quad 3(x-2\,y).$
7. $(2\,x-1)(x+4), \quad 2(x+4), \quad 4(2\,x-1).$
8. $4\,a^2+12\,ab+9\,b^2, \quad 2\,a+3\,b.$
9. $9\,x^2+24\,xy+16\,y^2, \quad 9\,x^2-16\,y^2.$
10. $x^2-10\,xy+25\,y^2, \quad x^2-25\,y^2.$

WRITTEN EXERCISES

Find the H. C. F. and also the L. C. M of each of the following sets:

1. $x^2+y^2, \; x^6+y^6.$
2. $x^2+xy+y^2, \; x^3-y^3.$
3. $x^2-5\,x-6, \quad x^2-2\,x-3, \quad x^2+19\,x+18.$
4. $x^4-6\,x^2+1, \quad x^3+x^2-3\,x+1, \quad x^3+3\,x^2+x-1.$
5. $162\,a^3b+252\,a^2b^2+98\,ab^3, \; 54\,a^3+42\,a^2b.$
6. $2\,x^3+x^2-8\,x+3, \quad x^2+2\,x-1.$
7. $3\,r^3+5\,r^2-7\,r-1, \quad 3\,r^2+8\,r+1.$
8. $a^3-3\,a^2+4, \quad ax-ab-2\,x+2\,b.$
9. $a^6+2\,a^3b^3+b^6-2\,a^4b-2\,ab^4, \quad a^3-2\,ab+b^3.$
10. $8\,a^3-36\,a^2b+54\,ab^2-27\,b^3, \quad 4\,a^2-9\,b^2.$
11. $2\,y^2+4\,by+3\,cy+6\,bc, \quad y^2-3\,by-10\,b^2.$
12. $x^{16}-y^{16}, \quad x^8-y^8, \quad x^4-y^4.$
13. $m^3+8\,m^2+7\,m, \quad m^3+3\,m^2-m-3, \quad m^3-7\,m-6.$

CHAPTER IV

FRACTIONS

74. An algebraic fraction is the indicated quotient of two algebraic expressions.

Thus $\frac{n}{d}$ means n *divided by* d.

In arithmetic a fraction such as $\frac{2}{3}$ is usually regarded as 2 of the 3 equal parts of a unit.

However, a fraction such as $\frac{5}{3\frac{1}{2}}$ cannot be regarded in this way, since a unit cannot be divided into $3\frac{1}{2}$ equal parts. $\frac{5}{3\frac{1}{2}}$ indicates that 5 is to be divided by $3\frac{1}{2}$; *i.e.* $\frac{5}{3\frac{1}{2}} = 5 \div 3\frac{1}{2}$.

Since a fraction is a quotient, it follows from the definition of division that if a fraction is multiplied by its denominator the product is the numerater.

That is, $$b \times \frac{a}{b} = a.$$

75. Fundamental Theorem on Fractions. *Both terms of a fraction may be divided or both may be multiplied by the same number without changing the value of the fraction.*

Proof. If a, b, and k are any numbers, we are to prove that

$$\frac{ak}{bk} = \frac{a}{b}.$$

From § 74, we have $$a = \frac{a}{b} \times b.$$

Multiplying both sides by k, $$ak = \frac{a}{b} \times bk.$$

Dividing both sides by bk, $$\frac{ak}{bk} = \frac{a}{b}. \qquad (1)$$

Reading (1) in the reverse order, we have

$$\frac{a}{b} = \frac{ak}{bk}. \qquad (2)$$

76. Reduction of Fractions. The form of a fraction may be modified in various ways without changing its value. Any such transformation is called a *reduction of the fraction.*

The most important reductions are the following:

(*A*) *Reduction by manipulation of signs.*

E.g. $\dfrac{n}{d}=-\dfrac{-n}{d}=-\dfrac{n}{-d}=\dfrac{-n}{-d}$; $\dfrac{b-a}{c-d}=-\dfrac{a-b}{c-d}=\dfrac{a-b}{d-c}$.

(*B*) *Reduction to lowest terms.*

E.g. $\dfrac{x^4+x^2+1}{x^6-1}=\dfrac{\cancel{(x^2+x+1)}\cancel{(x^2-x+1)}}{(x-1)\cancel{(x^2+x+1)}(x+1)\cancel{(x^2-x+1)}}=\dfrac{1}{(x-1)(x+1)}$.

(*C*) *Reduction to integral or mixed expressions.*

E.g. $\dfrac{2x^3+x^2+x+2}{x^2+1}=2x+1+\dfrac{-x+1}{x^2+1}=2x+1-\dfrac{x-1}{x^2+1}$.

(*D*) *Reduction to fractions having a common denominator.*

E.g. $\dfrac{2}{x+3}$ and $\dfrac{3}{x+2}$ become respectively $\dfrac{2(x+2)}{(x+3)(x+2)}$ and $\dfrac{3(x+3)}{(x+3)(x+2)}$; $a+1$ and $\dfrac{1}{a-1}$ become respectively $\dfrac{a^2-1}{a-1}$ and $\dfrac{1}{a-1}$.

77. These reductions are useful in connection with the various operations upon fractions. They depend upon the principles indicated below.

Reduction (*A*) is simply an application of the law of signs in division, § 22. It is often needed in connection with reduction (*D*). See p. 43.

Reduction (*B*) depends upon the theorem, § 75, $\dfrac{ak}{bk}=\dfrac{a}{b}$, by which *a common factor may be removed from both terms of a fraction.* This reduction is *complete* when the numerator and denominator have been divided by their H. C. F.

Reduction (*C*) is merely the process of performing the indicated division.

Reduction (*D*) depends upon the theorem of § 75, $\dfrac{a}{b}=\dfrac{ka}{kb}$, by which *a common factor is introduced into the terms of a fraction.*

A fraction is thus reduced to another fraction whose denominator is *any required multiple* of the given denominator.

Niels Henrick Abel (1802–1829) was a distinguished Norwegian mathematician.

Though he lived only twenty-seven years, Abel made contributions of far-reaching importance in mathematics. Not least noteworthy of these was his proof that literal equations of degree higher than the fourth cannot, in general, be solved by means of radicals.

His complete works were published in 1839.

ORAL EXERCISES

Reduce the following fractions to lowest terms.

1. $\dfrac{x^2+5x+6}{x^2+6x+9}$.

2. $\dfrac{x^2+3x+2}{x^2-4}$.

3. $\dfrac{a^2-b^2}{a^2+2ab+b^2}$.

4. $\dfrac{a^2+4ab+4b^2}{a^2-4b^2}$.

5. $\dfrac{a^3-b^3}{a^2-b^2}$.

6. $\dfrac{a^3-b^3}{a^2-2ab+b^2}$.

7. $\dfrac{a^3-b^3}{(a^2+ab+b^2)c}$.

8. $\dfrac{(x+4)^2}{x^2+7x+12}$.

9. $\dfrac{(x-3)^2}{x-5x+6}$.

WRITTEN EXERCISES

Reduce the following so that the letters in each factor shall occur in alphabetical order, and no negative sign shall stand before a numerator or denominator, or before the first term of any factor.

1. $\dfrac{n-m}{b-a}$.

2. $-\dfrac{(b-a)(c-d)}{x(s-r-t)}$.

3. $\dfrac{-(x-y)}{(b-a)(c-d)}$.

4. $\dfrac{-(x-y)(z-y)}{-(b-a)(c-d)}$.

5. $\dfrac{r-s}{(a-b)(c-b)(c-a)}$.

6. $\dfrac{-a(c+b)}{b(c-a)}$.

7. $\dfrac{-(c-a)(d-c)}{(a-b)(b-c)}$.

8. $\dfrac{(b-a)(c-b)(c-a)}{(y-x)(y-z)(z-x)}$.

9. $-\dfrac{-1}{(a-b)(b-c)(c-a)}$.

10. $\dfrac{(c-b-a)(b-a-c)}{3(a-c)(b-c)(c-a)}$.

Reduce each of the following to lowest terms:

11. $\dfrac{a^4-b^4}{a^6-b^6}$.

12. $\dfrac{c^2-(a-b)^2}{(a+c)^2-b^2}$.

13. $\dfrac{7ax^2-56a^4x^5}{28x^2(1-64a^6x^6)}$.

14. $\dfrac{m^3+5m^2+7m+3}{m^2+4m+3}$.

15. $\dfrac{a^3-7a+6}{a^3-7a^2+14a-8}$.

16. $\dfrac{x^3+2x^2+2x+1}{x^4+x^3-x^2-2x-2}$.

17. $\dfrac{2x^3-x^2-8x-3}{2x^3-3x^2-7x+3}$.

18. $\dfrac{4x^3+8x^2-3x+5}{6x^3-5x^2+4x-1}$.

Reduce each of the following to an integral or mixed expression:

19. $\frac{x^4+1}{x+1}$. 21. $\frac{x^4}{x-1}$. 23. $\frac{c^5}{c^3+c^2-c+1}$.

20. $\frac{x^5+1}{x+1}$. 22. $\frac{a^3}{a^2+a+1}$. 24. $\frac{x^2-x+1}{x^2+x+1}$.

25. $\frac{a^4+a^2b^2+b^4}{a-b}$. 27. $\frac{x^3-x^2-x+1}{x^3+x^2+x-1}$.

26. $\frac{3a^3-3a^2+3a-1}{a-2}$. 28. $\frac{4m^4-3m^3+3}{2m^2-2m+1}$.

Reduce each of the following sets of expressions to equivalent fractions having the lowest common denominator:

29. $\frac{1}{x^4-3x^2y^2+y^4}$, $\frac{1}{x^2-xy-y^2}$, $\frac{1}{x^2+xy-y^2}$.

30. $\frac{x^2+y^2}{x^3+y^3+x^2-xy+y^2}$, $\frac{x+y-1}{x^2-xy+y^2}$, $\frac{x^2+xy+y^2}{x+y+1}$.

31. $\frac{x}{(a-b)(c-b)(c-a)}$, $\frac{y}{(a-b)(b-c)(a-c)}$,

32. $\frac{a}{b+c}$, $\frac{b}{c+a}$, $\frac{c}{a+b}$, d. $\left[\frac{z}{(b-a)(b-c)(a-c)}\right.$

33. $\frac{b-c}{(a-c)(a-b)}$, $\frac{a-b}{(c-a)(b-c)}$, $\frac{c-a}{(b-a)(c-b)}$.

34. $\frac{m-n}{a^3-6a^2+11a-6}$, $\frac{a+2}{a^2-4a+3}$, $\frac{a+3}{a^2-3a+2}$.

If a, b, m are positive numbers, arrange each of the following sets in decreasing order. Verify the results by substituting convenient numbers for a, b, m.

Suggestion. Reduce the fractions in each set to equivalent fractions having a common denominator.

35. $\frac{a}{a+1}$, $\frac{2a}{a+2}$, $\frac{3a}{a+3}$. 36. $\frac{m}{2m+1}$, $\frac{2m}{3m+2}$, $\frac{3m}{4m+3}$.

37. $\frac{a+3b}{a+4b}$, $\frac{a+b}{a+2b}$, $\frac{a+4b}{a+5b}$.

ADDITION AND SUBTRACTION OF FRACTIONS

78. Common Denominator. Fractions which have a common denominator are added or subtracted in accordance with the distributive law for division, § 29.

That is, $$\frac{a}{d}+\frac{b}{d}-\frac{c}{d}=\frac{a+b-c}{d}.$$

In order to add or subtract fractions not having a common denominator, they should first be reduced to *equivalent fractions having a common denominator.*

In manipulating fractions it is advantageous to keep their terms *in the factored form* as long as possible. When several fractions are to be combined, it is sometimes best to take only part of them at a time.

Example. $$\frac{1}{(x-1)(x-2)}-\frac{1}{(2-x)(x-3)}+\frac{1}{(3-x)(4-x)}.$$

Taking the first two together, we have

$$\frac{1}{(x-1)(x-2)}-\frac{1}{(2-x)(x-3)}=\frac{1}{(x-1)(x-2)}+\frac{1}{(x-2)(x-3)}$$
$$=\frac{2x-4}{(x-1)(x-2)(x-3)}=\frac{2}{(x-1)(x-3)}.$$

Taking this result with the third fraction,

$$\frac{2}{(x-1)(x-3)}+\frac{1}{(x-3)(x-4)}=\frac{3x-9}{(x-1)(x-3)(x-4)}=\frac{3}{(x-1)(x-4)}.$$

If all are taken at once, the work should be carried out as follows:

The numerator of the sum is

$$(x-3)(x-4)+(x-1)(x-4)+(x-1)(x-2).$$

Adding the first two terms with respect to $(x-4)$, we have

$$(x-3)(x-4)+(x-1)(x-4)=(2x-4)(x-4)=2(x-2)(x-4).$$

Again adding with respect to $x-2$ we have

$$2(x-2)(x-4)+(x-1)(x-2)=(3x-9)(x-2)=3(x-3)(x-2).$$

Hence, the sum is $$\frac{3(x-3)(x-2)}{(x-1)(x-2)(x-3)(x-4)}=\frac{3}{(x-1)(x-4)}.$$

WRITTEN EXERCISES

Perform the following indicated additions and subtractions:

1. $\dfrac{2}{x-3}+\dfrac{3}{x-4}-\dfrac{4}{x-5}$.

2. $\dfrac{1}{x-1}-\dfrac{4}{x-2}+\dfrac{7}{x-3}$.

3. $\dfrac{2}{(x+1)^2}+\dfrac{3}{x+1}+\dfrac{4}{x-2}$.

4. $\dfrac{1}{3(x+1)}-\dfrac{x-2}{3(x^2-x+1)}$.

5. $\dfrac{5x+6}{x^2+x+1}-\dfrac{3x-4}{x^2-x+1}$.

6. $\dfrac{3}{4(x+3)}-\dfrac{5}{8(x+5)}-\dfrac{1}{8(x+1)}$.

7. $\dfrac{1}{12(x+1)}-\dfrac{7}{3(x-2)}+\dfrac{13}{4(x-3)}$.

8. $\dfrac{1}{5(x+2)}+\dfrac{4x-8}{5(x^2+1)}$.

9. $\dfrac{2}{(x-2)^2}-\dfrac{1}{x-2}+\dfrac{1}{x+1}$.

10. $\dfrac{2}{(x-2)^2}+\dfrac{1}{5(x-2)}-\dfrac{x+2}{5(x^2+1)}$.

11. $\dfrac{1}{(x-1)^2}+\dfrac{1}{x-1}-\dfrac{1}{x^2-1}$.

12. $\dfrac{1}{2(1-3x)^3}+\dfrac{3}{8(1-3x)^2}+\dfrac{3}{32(1-3x)}+\dfrac{1}{32(1+x)}$.

13. $\dfrac{1}{(1-a)(2-a)}-\dfrac{1}{(2-a)(a-3)}+\dfrac{2}{(3-a)(a-1)}$.

14. $\dfrac{xy}{(z-y)(x-z)}-\dfrac{yz}{(x-z)(x-y)}-\dfrac{xz}{(y-x)(y-z)}$.

15. $\dfrac{1}{a-1}-\dfrac{2a-5}{a^2-2a+1}-\dfrac{5a^2-3a-2}{(a-1)^3}$.

16. $\dfrac{1}{m^2+m+1}-\dfrac{1}{m^2-m+1}+\dfrac{2m+2}{m^4+m^2+1}$.

17. $\dfrac{1}{b^2-3b+2}+\dfrac{1}{b^2-5b+6}-\dfrac{2}{b^2-4b+3}$.

18. $\dfrac{r+s}{(r-t)(s-t)}-\dfrac{s+t}{(r-s)(t-r)}-\dfrac{r+t}{(t-s)(s-r)}$.

19. $\dfrac{p^2+q^2}{(p-q)(p+r)}+\dfrac{q^2-pr}{(q-r)(q-p)}+\dfrac{r^2+pq}{(r-q)(r+p)}$.

MULTIPLICATION AND DIVISION OF FRACTIONS

79. Theorem. *The product of two fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

That is,
$$\frac{a}{b} \times \frac{n}{d} = \frac{an}{bd}.$$

For, let
$$x = \frac{a}{b} \times \frac{n}{d}.$$

Multiplying both sides by bd, $bdx = bd\left(\frac{a}{b} \times \frac{n}{d}\right)$.

By the commutative law of factors (§ 26), $bdx = b \times \frac{a}{b} \times d \times \frac{n}{d}$.

Canceling,
$$bdx = b \times \frac{a}{b} \times d \times \frac{n}{d} = an.$$

Dividing both sides by bd,
$$x = \frac{an}{bd}.$$

Therefore,
$$\frac{a}{b} \times \frac{n}{d} = \frac{an}{bd}.$$

80. Power of a Fraction. It follows that *a fraction is raised to any power by raising the numerator and denominator separately to that power.*

For
$$\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}, \quad \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}, \text{ etc.}$$

81. Reciprocal of a Fraction. A fraction multiplied by itself inverted equals $+1$.

For
$$\frac{n}{d} \times \frac{d}{n} = \frac{nd}{nd} = +1 \quad \text{and} \quad -\frac{n}{d} \times \left(-\frac{d}{n}\right) = \frac{nd}{nd} = +1.$$

If the product of two numbers is 1, each is called the *reciprocal* of the other. Hence the reciprocal of a fraction is the fraction inverted.

Also, since from $ab = 1$, we have $a = \frac{1}{b}$, and $b = \frac{1}{a}$, it follows that if two numbers are reciprocals of each other, then either is the quotient obtained by dividing 1 by the other.

82. Theorem. *Dividing by any number is equivalent to multiplying by its reciprocal.*

For $$n \div d \text{ or } \frac{n}{d} = n \times \frac{1}{d}.$$

83. Theorem. *Dividing a number by a fraction is equivalent to multiplying by the fraction inverted.*

For by § 81, the reciprocal of the fraction is the fraction inverted.

84. Theorem. *A fraction is divided by an integer by multiplying its denominator or dividing its numerator by that integer.*

For (1) $\frac{n}{d} \div a = \frac{n}{d} \times \frac{1}{a}$ by § 82,

but $\frac{n}{d} \times \frac{1}{a} = \frac{n}{ad}$ by § 79,

and (2) $\frac{n}{ad} = \frac{n \div a}{ad \div a} = \frac{n \div a}{d}$ by § 75.

In multiplying and dividing fractions, their terms should at once be put into factored forms.

85. Preliminary Reductions. When mixed expressions or sums of fractions are to be multiplied or divided, these operations are indicated by means of parentheses, and the additions or subtractions within the parentheses should be performed first.

Example 1. $$\left(1 + \frac{a}{b}\right) \times \left(1 - \frac{b}{a}\right) = \frac{b + a}{b} \times \frac{a - b}{a} = \frac{a^2 - b^2}{ab}.$$

Example 2. Simplify

$$\left[\left(1 - a + \frac{2a^2}{1 + a}\right) \div \left(\frac{1}{1 + a} - \frac{1}{1 - a}\right)\right] \times \frac{3a^3}{a^4 - 1}.$$

Performing the indicated operations within the parentheses, we have

$$\left[\frac{1 - a^2 + 2a^2}{1 + a} \div \frac{1 - a - 1 - a}{1 - a^2}\right] \times \frac{3a^3}{a^4 - 1} = \left[\frac{1 + a^2}{1 + a} \div \frac{2a}{a^2 - 1}\right] \times \frac{3a^3}{a^4 - 1}$$

$$= \frac{1 + a^2}{1 + a} \times \frac{a^2 - 1}{2a} \times \frac{3a^3}{(a^2 - 1)(a^2 + 1)} = \frac{3a^2}{2(a + 1)}.$$

WRITTEN EXERCISES

Perform the following indicated operations and reduce each result to its simplest form.

1. $$\frac{x^4+x^2y^2+y^4}{x^3-y^3}\times\frac{x^2-y^2}{x^3+y^3}.$$

2. $$\frac{a^2-b^2x^2+acx^2-bcx^3}{(6a^2+3a-4)(6a^2-3a+4)}\div\frac{-ay^2-bxy^2-cx^2y^2}{20x^2-15ax^2-30a^2x^2}.$$

3. $$\frac{20r^2s^2+23rst-21t^2}{8m^2n^3-48m^2n^2y+72m^2ny^2}\times\frac{12mn^3-28mn^2y-24mny^2}{10r^2s^2+24rst-18t^2}\left[\div\frac{3n+2y}{2s+3t}.\right.$$

4. $$\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}-\frac{a}{b}-\frac{b}{a}+1\right)\div\frac{a^5+b^5}{a-b}.$$

5. $$\left(\frac{1}{x}+\frac{1}{y}\right)\left(\frac{1}{x}-\frac{1}{y}\right)\left(1-\frac{x-y}{x+y}\right)\left(2+\frac{2y}{x-y}\right).$$

6. $$\left(\frac{a}{a-b}-\frac{b}{a+b}\right)\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\div\left(\frac{1}{a^2}+\frac{1}{b^2}\right).$$

7. $$\left(1+\frac{b}{a-b}\right)\left(1-\frac{b}{a+b}\right)\div\left(1+\frac{b^2}{a^2-b^2}\right).$$

8. $$\left(\frac{m+n}{m-n}-\frac{m-n}{m+n}\right)\left(m+n+\frac{2n^2}{m-n}\right)\div\left(\frac{m+n}{m-n}+\frac{m-n}{m+n}\right).$$

9. $$\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right)\cdot\left(x^2+y^2+\frac{2x^2y^2+2y^4}{x^2-y^2}\right)\div\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right).$$

10. $$\left(\frac{x+y+z}{x+y}+\frac{z^2}{(x+y)^2}\right)\cdot\left(\frac{(x+y)^3}{(x+y)^3-z^3}\right)\cdot\left(1+\frac{z}{x+y}\right).$$

11. $$\frac{a^2+ab+b^2}{a^2-ab+b^2}\cdot\frac{a+b}{a^3-b^3}\cdot\left(a^2+\frac{b^3-a^2b}{a+b}\right).$$

12. $$\frac{m^2+mn}{m^2+n^2}\cdot\frac{m^3-mn^2-m^2+n^2}{m^3n-n^4}\cdot\frac{m^2n^2+mn^3+n^4}{m^4-2m^3+m^2}.$$

13. $$\left(xy^9+x^9y-\frac{2x^{11}y^3}{x^2y^2}\right)\div\left[\frac{x^2+y^2}{x^2}\cdot\left(\frac{1}{y^2}-\frac{1}{z^2}\right)-\frac{x^2+y^2}{y^2}\cdot\left(\frac{1}{x^2}-\frac{1}{z^2}\right)\right].$$

COMPLEX FRACTIONS

86. **A Complex Fraction** is one which contains a fraction either in its numerator or in its denominator or in both.

Since every fraction is an indicated division, any complex fraction may be simplified by performing the division.

It is usually better, however, to remove all the minor denominators at once *by multiplying both terms of the complex fraction by the least common multiple of all the minor denominators according to the fundamental theorem* § 75.

For example, $$\frac{\dfrac{x}{3}+\dfrac{x}{2}}{\dfrac{2x^2}{3}-\dfrac{3}{2}}=\frac{\left(\dfrac{x}{3}+\dfrac{x}{2}\right)\cdot 6}{\left(\dfrac{2x^2}{3}-\dfrac{3}{2}\right)\cdot 6}=\frac{2x+3x}{4x^2-9}=\frac{5x}{4x^2-9}.$$

Again in $\dfrac{\dfrac{2}{x-1}-\dfrac{1}{x-2}}{\dfrac{3}{x-4}-\dfrac{4}{x-1}}$ multiply both terms by $(x-1)(x-2)(x-4)$, obtaining $$\frac{2(x-2)(x-4)-(x-1)(x-4)}{3(x-1)(x-2)-4(x-2)(x-4)}$$

$$=\frac{2x^2-12x+16-x^2+5x-4}{3x^2-9x+6-4x^2+24x-32}=-\frac{x^2-7x+12}{x^2-15x+26}.$$

A complex fraction may contain another complex fraction in one of its terms.

E.g. $\dfrac{1}{a+\dfrac{a+1}{a+\dfrac{1}{a-1}}}$ has the complex fraction $\dfrac{a+1}{a+\dfrac{1}{a-1}}$

in its denominator. This latter fraction is first reduced by multiplying its numerator and denominator by $a-1$, giving

$$\frac{1}{a+\dfrac{a+1}{a+\dfrac{1}{a-1}}}=\frac{1}{a+\dfrac{a^2-1}{a^2-a+1}}=\frac{a^2-a+1}{a^3+a-1}.$$

WRITTEN EXERCISES

Simplify each of the following:

1. $$\frac{\dfrac{m^2+mn}{m^2-n^2}}{\dfrac{m}{m-n}-\dfrac{n}{m+n}}.$$

2. $$\frac{\dfrac{a^4-b^4}{a^2-2ab+b^2}}{\dfrac{a^2+ab}{a-b}}.$$

3. $$\frac{\dfrac{x^5-3x^4y+3x^3y^2-x^2y^3}{x^3y-y^4}}{\dfrac{x^5-2x^4y+x^3y^2}{x^2y^2+xy^3+y^4}}.$$

4. $$\frac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2-x^2}}{\dfrac{1}{a+x}-\dfrac{1}{a-x}-\dfrac{2a}{a^2-x^2}}.$$

5. $$\frac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2+x^2}}{\dfrac{1}{a-x}-\dfrac{1}{a+x}+\dfrac{2x}{a^2+x^2}}.$$

6. $$\frac{m^2-mn+n^2-\dfrac{m^3-n^3}{m+n}}{m^2+mn+n^2+\dfrac{m^3+n^3}{m-n}}.$$

7. $$\frac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}.$$

8. $$\frac{\dfrac{a-\dfrac{1}{a^2}}{a-2+\dfrac{1}{a}}}{\dfrac{a^2+1+\dfrac{1}{a^2}}{a-2+\dfrac{2}{a}-\dfrac{1}{a^2}}}.$$

9. $$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}.$$

10. $$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}}.$$

11. $$\frac{3}{3+\dfrac{3}{3+\dfrac{3}{3+\dfrac{3}{x}}}}.$$

12. $$\frac{\dfrac{a}{a+b}+\dfrac{b}{a-b}+\dfrac{2ab}{a^2-b^2}}{\dfrac{1}{(a+b)^2}+\dfrac{1}{(a-b)^2}}.$$

CHAPTER V

EQUATIONS OF THE FIRST DEGREE IN ONE UNKNOWN

87. Rational Equations. An equation *is rational in a given letter* if every term in the equation is rational with respect to that letter. See § 15.

E.g. $3x + 2 = x - 4$ is rational in x,

as is also $\frac{2}{x} + 5 = \frac{3}{x}$.

But $\sqrt{x} + 2 = x + 3$ is irrational in x.

88. Integral Equations. An equation is *integral in a given letter* if every term is rational and integral in that letter; it is fractional if it contains the letter in any denominator.

E.g. $x - 2 = 2x + 3$ is integral in x,

while $\frac{2}{x} + 5 = \frac{3}{x}$ is fractional in x.

89. Degree of an Equation. The *degree* of a rational, integral equation in a given letter is the highest exponent of that letter in the equation.

E.g. $2x - 3 = 5$ is of the first degree in x,

and $x^2 + 3x = 4$ is of the second degree in x.

In determining the degree of an equation according to this definition it is necessary that all indicated multiplications be performed as far as possible.

E.g. $(x-2)(x-3) = 0$ is of the second degree in x, since it reduces to $x^2 - 5x + 6 = 0$.

90. Substitution. When in an algebraic expression a letter is replaced by another number symbol, this is called a *substitution on that letter.*

E g. In the expression $2a + 5$, if a is replaced by 3, giving $2 \cdot 3 + 5$, this is a substitution on the letter a.

91. Satisfying an Equation. An equation containing a single letter is said to be *satisfied* by any substitution on that letter which reduces both members of the equation to the same number.

E.g. $4x + 8 = 24$ is satisfied by $x = 4$, since $4 \cdot 4 + 8 = 24$.

92. Identity. An equation in a single letter which is satisfied by *every substitution* on that letter is called an *identity*.

E.g. $2(x + 3) = 2x + 6$ is an identity.

If an equation is an identity, *both members reduce to the same expression* when all indicated operations are performed.

E.g. In $2(x + 3) = 2x + 6$, both members are $2x + 6$ when the operations indicated are performed.

93. Equation of Condition. An equation which is not an identity is called an *equation of condition* or simply an *equation.*

The members of an equation of condition *cannot be reduced to the same expression* by performing the indicated operations.

E.g. $3(x - 2) = 4(x - 3)$ cannot be so reduced. This equation is satisfied by $x = 6$, and by no other value of x.

94. Root of an Equation. A value of the unknown which satisfies an equation is called a *root* or *solution* of the equation.

E.g. $x = 6$ is a root of the equation $3(x - 2) = 4(x - 3)$.

ORAL EXERCISES

1. What is the value of $3x^2 - 2x + 1$ if 2 is substituted for x?

2. Is $x^2 + 4x + 4 = (x + 2)^2$ an identity or simply an equation?

3. What is the degree of the equation $(x-1)(x-2)(x-3) = 0$?

4. Is the equation $\frac{5x-1}{4} + \frac{2x^2+7}{6} = 0$ integral or fractional in x? Why?

5. Is $\frac{2x+5}{x-1} + \frac{3}{x} + 16 = 0$ integral or fractional in x?

WRITTEN EXERCISES

1. Find whether $4(x-1)=3(x^2-1)$ is satisfied by $x=1$; by $x=2$; by $x=\frac{1}{3}$.

2. Is $3a+4b=12$ satisfied by $a=0,\ b=3$? by $a=4$, $b=0$? by $a=2,\ b=2$?

3. Is $x^3+3x^2+3x+1=(x+1)^3$ satisfied by $x=1$? by $x=2$? by $x=-1$? by $x=-2$?

What kind of an equation is this?

4. Find by performing the indicated operations whether or not the following is an identity:

$$12(x+y)^2+17(x+y)-7=(3x+3y-1)(4x+4y+7).$$

5. Find whether the following is an identity:

$$2(a+b)^2+5(a+b)+8ab=(2a+2b+1)(a+b+1).$$

6. Is $x^2-16=(x-4)(x+5)(x+6)$ satisfied by $x=2$? by $x=3$? by $x=4$?

SOLUTION OF EQUATIONS

95. Equivalent Equations. Two equations are said to be equivalent *if every root of either of them is also a root of the other.*

E.g. $3x+6=12$ and $x+2=4$ are both satisfied by $x=2$ and by no other values of x. Hence, these equations are equivalent.

96. Axioms. The process of solving equations is based on the following *axioms:*

Axiom 1. *If equals are added to equals, the sums are equal.*

Axiom 2. *If equals are subtracted from equals, the remainders are equal.*

Axiom 3. *If equals are multiplied by equals, the products are equal.*

Axiom 4. *If equals are divided by equals, the quotients are equal.*

Axiom 5. *If two algebraic expressions are equal to the same expression, they are equal to each other.*

Axiom 6. *Any quantity may be substituted for its equal.*

97. Deriving Equivalent Equations. From these axioms it follows that an equation may be changed into an equivalent equation by any one of the following processes:

(1) *Adding the same number to both members.*

(2) *Subtracting the same number from both members.*

(3) *Multiplying both members by any known number not zero.*

(4) *Dividing both members by any known number not zero.*

(5) *Changing the form of either member in any way which leaves its value unaltered.*

98. Multiplication by zero is always *a possible operation*, but the equation resulting from multiplying both members by zero is not equivalent to the original equation.

Thus, $x + 4 = 8$ is satisfied by $x = 4$, and by no other number. But $(x + 4) \cdot 0 = 8 \cdot 0$ reduces to $0 = 0$ no matter what value is given to x.

Hence, in solving an equation, *its members must never be multiplied by an expression which equals zero.*

99. Division by zero is always an *impossible operation* or an *indeterminate operation* (see § 31), and hence *the members of an equation should never be divided by any expression which equals zero.*

100. Solving an Equation. The process of solving an equation consists in deriving successive equivalent equations, each simpler than the preceding, until finally one is reached in which the unknown stands alone on one side, and does not occur on the other side.

Example 1.

$$\text{Solve } 2(4x + 5) + 18x + 49 = 7(6 + 6x) + 2x + 8. \qquad (1)$$

Solution. Performing indicated operations, that is, changing the form of each member but not its value,

$$8x + 10 + 18x + 49 = 42 + 42x + 2x + 8. \qquad (2)$$

Performing further indicated operations,

$$26x + 59 = 44x + 50. \qquad (3)$$

Subtracting $26x$ and also 50 from each member,

$$9 = 18x. \qquad (4)$$

Dividing both members by 18,

$$x = \tfrac{9}{18} = \tfrac{1}{2}. \qquad (5)$$

Example 2. Solve, $(3x-2)(x+4)-7=(x-1)(2x-3)$
$+x^2+12.$

Solution. Performing indicated operations,

$$\cancel{3x^2}+10x-8-7=\cancel{2x^2}-5x+3+\cancel{x^2}+12.$$

Subtracting $3x^2$ from both members and adding $5x$ and 15 to both members, we have

$$15x=30.$$

Dividing by 15, $x=2.$

Check. Substituting $x=2$,

$$(3\cdot 2-2)(2+4)-7=(2-1)(2\cdot 2-3)+2^2+12$$
$$24-7=1+4+12$$
$$17=17.$$

It should be noted that, in case an integral equation is multiplied by an expression containing the unknown, any value of the unknown which makes the multiplier zero is a root of the resulting equation.

Thus, if $x+4=8$ is multiplied by $x-2$ giving $(x+4)(x-2)=8(x-2)$, a new root $x=2$ is introduced.

Similarly, if a factor containing the unknown is removed from both members of an equation, then any value which makes this factor zero is a root of the original equation but may not be a root of the resulting equation.

Hence, multiplying by an expression containing the unknown introduces new roots, while dividing by such an expression removes roots.

ORAL EXERCISES

Solve the following equations:

1. $x+7=15.$
2. $x-3=-8.$
3. $13x=48+x.$
4. $5x=3+4x.$
5. $10y-4=9y+6.$
6. $42w=20w+66.$
7. $10m+18=2m+50.$
8. $25x-2=8x+66.$
9. $20x-30=-7x-4.$
10. $-8x=-33+3x.$
11. $-4a+45=7-2a.$
12. $-3y+6=-4y-2.$
13. $-2x-3=2x+5.$
14. $5x-2=-7x+5.$
15. $-5z-4=-3z+2.$
16. $-x-8=-11x+2.$
17. $4x+7=-2x-5.$
18. $2(x-3)=3(x+2).$
19. $3(x-2)=4(x-1).$
20. $-2(2-x)=5(x+1).$

WRITTEN EXERCISES

Solve the following equations:

1. $7(x+6)+10x=5x+66.$
2. $6x+8(x+1)=89-3(2x+7).$
3. $4(3x+2)=18x+36-5(9x+3).$
4. $42(1+x)-2(2x+5)=16x-34.$
5. $17(x-1)+20=-3(2x-1).$
6. $(2a+3)(3a-2)=a^2+a\,5a+3).$
7. $6(b-4)^2=-5-(3-2b)^2-5(2+b)(7-2b).$
8. $(y-3)^2+(y-4)^2-(y-2)^2-(y^2-9)=0.$
9. $(x-3)(3x+4)-(x-4)(x-2)=(2x+1)(x-6).$
10. $2(3r-2)(4r+1)+(r-4)^3=(r+4)^3-2.$
11. $a^3-c+b^3c+abc=b.$ (Solve for c).

In the next three examples solve for y:

12. $(b-2)^2(b-y)-3by+(2b+1)(b-1)=3-2b.$
13. $ny(y+n)-(y+m)(y+n)(m+n)+my(y+m)=0.$
14. $(m+n)(n+b-y)+(n-m)(b-y)=n(m+b).$

In the next eight examples solve for x:

15. $2(12-x)+3(5x-4)+2(16-x)=12(3+x).$
16. $(b-a)x-(a+b)x+4a^2=0.$
17. $(x-a)(b-c)+(b-a)(x-c)-(a-c)(x-b)=0.$
18. $(x-3)(x-7)-(x-5)(x-2)+12=2(x-1).$
19. $(a+b)^2+(x-b)(x-a)-(x+a)(x+b)=0.$
20. $\frac{7}{12}(5x-1)+\frac{5}{16}(2-3x)+\frac{1}{3}(4+x)=\frac{3}{8}(1+2x)-\frac{9}{16}.$
21. $a(x-b)-(a+b)(x+b-a)=b(x-a)+a^2-b^2.$
22. $(l-m)(x-n)+2l(m+n)=(l+m)(x+n).$

Solve each of the following equations for each letter in terms of the others:

23. $l(W+w')=l'W'.$
24. $(v-n)d=(v-n_1)d_1.$
25. $m_2s_2(t_2-t)=(m+m_1)(t-t_1).$
26. $(m+m_1)(t_1-t)=lm_2+m_2t.$

PROBLEMS

1. What number must be added to each of the numbers 2, 26, and 10 in order that the product of the first two sums may equal the square of the last sum?

2. What number must be subtracted from each of the numbers 9, 12, and 18 in order that the product of the first two remainders may equal the square of the last remainder?

3. What number must be added to each of the numbers a, b, c in order that the product of the first two sums may equal the square of the last?

Note that problem 1 is a special case of 3.

4. What number must be added to each of the numbers 4, 2, 3, and 1 in order that the product of the first two sums may equal the product of the last two?

5. What number must be added to each of the numbers a, b, c, d in order that the product of the first two sums may equal the product of the last two?

6. What number must be added to each of the numbers 3, 1, 5, 2 in order that the sum of the squares of the first two sums may equal the sum of the squares of the last two?

7. What number must be added to each of the numbers a, b, c, d in order that the sum of the squares of the first two sums may equal the sum of the squares of the last two?

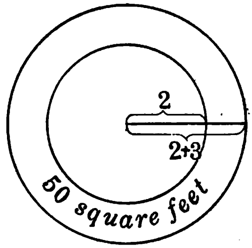
Note that problem 6 is a special case of problem 7.

8. What number must be added to each of the numbers 3, 6, 9, 7 in order that the sum of the squares of the first two sums may be 4 more than twice the product of the last two?

9. What number must be added to each of the numbers a, b, c, d in order that the sum of the squares of the first two sums may be k more than twice the product of the last two?

10. The radius of a circle is increased by 3 feet, thereby increasing the area of the circle by 50 square feet. Find the radius of the original circle.

The area of a circle is πr^2. Use $3\frac{1}{7}$ for π. Hence $(r+3)^2 \cdot \frac{22}{7} = \frac{22}{7} r^2 + 50$.



11. The radius of a circle is decreased by two feet, thereby decreasing the area by 36 square feet. Find the radius of the original circle.

12. The radius of a circle is increased by a feet, thereby increasing the area by b square feet. Find the radius of the original circle.

13. The radius of a circle is decreased by a feet, thereby decreasing the area by b square feet. Find the radius of the original circle.

14. Each side of a square is increased by 2 feet, thereby increasing its area by 12 square feet. Find the side of the original square.

15. Each side of a square is decreased by 3 feet, thereby decreasing its area by 15 square feet. Find the side of the original square.

16. Each side of a square is increased by a feet, thereby increasing its area by b square feet. Find the side of the original square.

17. Two opposite sides of a square are each increased by 5 feet and the other two sides by 7 feet, thereby producing a rectangle whose area is 90 square feet greater than that of the square. Find the side of the square.

18. Two opposite sides of a square are each decreased by 2 feet, the other two sides by 3 feet, thereby producing a rectangle whose area is 60 square feet less than that of the square. Find the side of the square.

19. Two opposite sides of a square are each increased by a feet and the other two by b feet, thereby producing a rectangle whose area is c square feet greater than that of the square. Find the side of the square.

20. A messenger starts for a distant point at 4 A.M., going 5 miles per hour. Four hours later another starts from the same place, going in the same direction at the rate of 9 miles per hour. When will they be together? When will they be 8 miles apart? How far apart will they be at 2 P.M.?

21. One object moves with a velocity of v_1 feet per second and another along the same path in the same direction with a velocity of v_2 feet. How long will it require the latter to gain n feet on the former?

Let $t =$ the required number of seconds. Then the first object moves v_1t feet and the second object moves v_2t feet in the required time.

Hence, $$v_2t = v_1t + n,$$

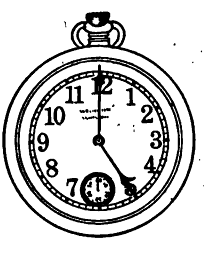
and $$t = \frac{n}{v_2 - v_1}.$$

Discussion. If $v_2 > v_1$ and $n > 0$, the value of t is positive, *i.e.* the objects will be in the required position some time *after* the time of starting.

If $v_2 < v_1$ and $n > 0$, the value of t is negative, which may be taken to mean that if the objects had been moving before the instant taken in the problem as the time of starting, then they would have been in the required position some time *earlier*.

If $v_2 = v_1$ and $n \neq 0$, the solution is impossible. See § 31. This means that the objects will never be in the required position.

22. State and solve a problem which is a special case of 21 under each of the conditions mentioned in the discussion.

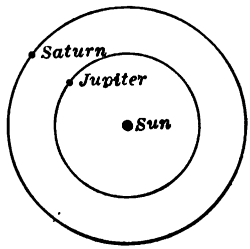


23. At what time after 5 o'clock are the hands of a clock first in a straight line?

24. How many minutes after 8 o'clock will the minute hand be 12 minute spaces behind the hour hand?

Two planets are said to be in conjunction when they are in a straight line with the sun. In the following problem conjunction means that the planets are on the *same side* of the sun.

25. Saturn completes its journey about the sun in 29 years and Jupiter in 12 years. How many years elapse from conjunction to conjunction?



26. An object moves in a fixed path at the rate of v_1 feet per second, and another which starts a seconds later moves in the same path at the rate of v_2 feet per second. In how many seconds will the latter overtake the former?

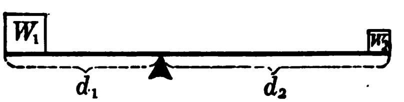
27. In problem 26 how long before they will be d feet apart?

If d is zero in problem 27, then it is the same problem as 26. If d is not zero and a is zero, it is the same as problem 21. Solve both of these problems, using the formula obtained in problem 27.

28. A, who weighs 75 pounds, sits 7 feet from the fulcrum. If B weighs 105 pounds, at what distance from the fulcrum should he sit in order to make a balance?

Suggestion. **If the distances from the ends of a lever to the fulcrum are d_1 and d_2 and if the weights on the lever at its ends are w_1 and w_2, then**

$$d_1w_1 = d_2w_2.$$



29. A and B together weigh $212\frac{1}{2}$ pounds. They balance when A is 6 feet, and B is $6\frac{3}{4}$ feet, from the fulcrum. Find the weight of each.

30. A lever 9 feet long carries weights of 17 and 32 pounds respectively at its ends. Where should the fulcrum be placed so as to make the lever balance?

31. A lever of unknown length is balanced when weights of 30 and 45 pounds respectively, are placed on it at opposite ends. Find the length of the lever, if the smaller weight is two feet farther from the fulcrum than the greater.

Suggestion. Let x be the distance from the greater weight to the fulcrum.

32. A beam carries 3 weights, one at each end weighing 100 and 120 pounds respectively, and the third weighing 150 pounds 2 feet from its center, where the fulcrum is. What is the length of the beam if this arrangement makes it balance?

33. A beam whose fulcrum is at its center is made to balance when weights of 60 and 80 pounds are placed at one end and 2 feet from that end respectively, and weights of 50 and 100 pounds are placed at the other end and 3 feet from it respectively. Find the length of the beam.

34. A man weighing 190 pounds is trying to pry up a rock by use of a plank 12 feet long and a block on which the plank rests as a fulcrum. Find the weight of the stone if he can just lift it when the fulcrum is 2 feet from the stone. How heavy a stone could he lift by putting the fulcrum 3 feet away from it? Four feet away from it?

35. Two automobiles are racing on a circular track. One makes the circuit in 31 minutes and the other in $38\frac{1}{2}$ minutes. In what time will the faster machine gain 1 lap on the slower?

36. At what times between 12 o'clock and 6 o'clock are the hands of a watch together? (Find the time required to gain one circuit, two circuits, etc.)

37. The planet Mercury makes a circuit around the sun in 3 months and Venus in $7\frac{1}{2}$ months. Starting in conjunction, how long before they will again be in this position?

CHAPTER VI

FRACTIONAL EQUATIONS IN ONE UNKNOWN

101. Clearing of Fractions. When an equation contains fractions, the first step in the solution is usually to *clear it of fractions.* This may be done by multiplying both members of the equation by such an expression as *will cancel all the denominators,* namely, by the least common multiple of all the denominators.

E.g. In the equation $\frac{x}{x+1}=\frac{2}{x-1}+\frac{x+3}{x-1}$, if we multiply both members by x^2-1, all the denominators can be canceled.

Thus, $$\frac{x\overset{x-1}{\cancel{(x^2-1)}}}{\cancel{x+1}}=\frac{2\overset{x+1}{\cancel{(x^2-1)}}}{\cancel{x-1}}+\frac{(x+3)\overset{x+1}{\cancel{(x^2-1)}}}{\cancel{x-1}}.$$

Hence, $$x(x-1)=2(x+1)+(x+3)(x+1),$$

or $$\cancel{x^2}-x=2x+2+\cancel{x^2}+4x+3.$$

Transposing, $$-7x=5.$$

Hence, $$x=-\tfrac{5}{7}.$$

Check. Substituting $x=-\frac{5}{7}$, we get

$$-\tfrac{5}{2}=-\tfrac{14}{12}-\tfrac{16}{12}=-\tfrac{30}{12}=-\tfrac{5}{2}.$$

102. Checking Solutions. It does not always follow that the value of the unknown found by clearing of fractions is a *solution of the given equation.*

E.g. In the equation $\frac{x}{x-1}=\frac{1}{x-1}+\frac{x}{x+1}$, if we multiply both members by x^2-1, and cancel the denominators, we get

$$x(x+1)=(x+1)+x(x-1),$$

or $$\cancel{x^2}+x=x+1+\cancel{x^2}-x.$$

Hence, $$x=1.$$

But $x=1$, will not satisfy the given equation since it reduces the left member and also the first term in the right member to $\frac{1}{0}$, which is an impossible operation. (See § 31.) This equation, therefore, *has no solution.*

103. *It is therefore necessary to examine any result, obtained in solving an equation by a process which involves clearing of fractions, to see that no denominator in the given equation is reduced to zero when this result is substituted.*

We have, then, the following

Rule: To solve an equation containing fractions:

(1) *Reduce all fractions to their lowest terms.*

(2) *Multiply both members by the lowest common multiple of the denominators.*

(3) *Reject any root of the resulting equation which reduces any denominator of the given equation to zero. Any other solution thus obtained will satisfy the given equation.*

104. Fractional Equations Solved as Linear Equations. If when an equation is cleared of fractions, all terms cancel except those of the first degree, then the resulting equation is of the first degree, and may be solved by the methods heretofore used for such equations.

E.g. The two illustrations given on page 63 lead to first degree equations, since the terms containing x^2 cancel after clearing of fractions.

WRITTEN EXERCISES

Solve the following equations, rejecting all solutions which reduce any denominator to zero:

1. $\frac{4}{x+1}-\frac{3}{x-1}=\frac{-2}{x^2-1}$.

2. $\frac{1}{x}+\frac{2}{x-1}=\frac{8}{x(x-1)}$.

3. $\frac{1}{x-1}+\frac{1}{x+1}=\frac{4}{x^2-1}$.

4. $\frac{3}{x-3}-\frac{2}{x-2}=\frac{8}{4x^2-20x+24}$.

5. $\frac{1-2x}{3-4x}-\frac{5-6x}{7-8x}=\frac{8}{3}\cdot\frac{1-3x^2}{21-52x+32x^2}$.

6. $\frac{1+3x}{5+7x}-\frac{9-11x}{5-7x}=14\cdot\frac{(2x-3)^2}{25-49x^2}$.

7. $\frac{x-9}{x-5}-\frac{x-7}{x-2}-\frac{x-9}{x-4}=\frac{x-8}{x-5}-\frac{x-7}{x-4}-\frac{x-8}{x-2}.$

Suggestion. First transpose and unite the fractions having a common denominator.

8. $\frac{3x-4}{x+5}-\frac{4x-1}{x+4}=-\frac{x^2+44}{x^2+9x+20}.$

9. $\frac{1}{x-1}-\frac{2}{2x+1}=\frac{1}{x-2}-\frac{2}{2x-1}.$

10. $\frac{x-1}{x-2}+\frac{x-2}{3-x}=\frac{x-3}{x-4}-\frac{x-4}{x-5}.$

11. $\frac{7x+3}{5}-2=\frac{21x+9}{15}+\frac{17x-3}{3x+11}.$

12. $\frac{x-2}{x-3}-\frac{x-3}{x-2}=\frac{4x-9}{x^2-5x+6}.$

13. $\frac{x+1}{x+2}-\frac{x-1}{x+3}=\frac{2x+7}{x^2+5x+6}.$

14. $\frac{x-1}{x+3}-\frac{x+1}{x-3}=\frac{2x+1}{x^2-9}.$

15. $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}=\frac{4ab}{4b^2-x^2}.$

16. $\frac{1}{a-b}+\frac{a-b}{x}=\frac{1}{a+b}+\frac{a+b}{x}.$

17. $\frac{p^2x}{3(m+n)^2-px(m+n)}=\frac{p}{2(m+n)}.$

18. $\frac{m-q}{x-n}+\frac{n-p}{x-q}=\frac{m-q}{x-p}+\frac{n-p}{x-m}.$

Transpose and add the fractions with the same numerators.

19. $\frac{x+m}{x-m}-\frac{x-m}{x+m}=\frac{x+a}{x^2-m^2}.$

20. $\frac{3x-2}{2x+3}=\frac{x^2+15x+31}{2x^2+5x+3}+\frac{x-1}{x+1}.$

21. $\frac{2x-3}{2x+2}-\frac{x-8}{5x+2}=\frac{x+2}{2x+2}+\frac{3}{10}.$

22. $\frac{18x^2+7x-3}{9x^2-1}-\frac{3x+1}{3x-1}=1.$

23. $\frac{7x^2+11x+4}{6x^2+13x+5}+\frac{x+3}{2x+1}=\frac{5x+11}{3x+5}.$

24. $\frac{3x+1}{5x-7}-\frac{x-3}{2x-7}=\frac{x^2-10x+11}{10x^2-49x+49}.$

25. $a^2b-\frac{a+x}{b}=ab^2-\frac{b+x}{a}.$

PROBLEMS LEADING TO FRACTIONAL EQUATIONS

1. Find a number such that, if it is added to each term of the fraction $\frac{3}{8}$ and subtracted from each term of the fraction $\frac{13}{24}$, the results will be equal.

2. Find a number such that, if it is added to each term of the fraction $\frac{a}{b}$ and subtracted from each term of the fraction $\frac{m}{n}$, the results will be equal.

3. Find a number of two digits in which the tens' digit is 5 greater than the units' digit, and such that, if the number is divided by the sum of its digits, the quotient is 7 and the remainder is 3.

4. In a number of three digits the tens' digit is 3 greater than the units' digit, and the hundreds' digit is 2 greater than the tens' digit. If the number is divided by the sum of the digits, the quotient is 50, and the remainder is 13. Find the number.

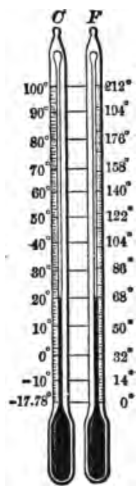
5. A pipe can fill a cistern in 6 hours, another in 9 hours, and a third can empty the cistern in 12 hours. How long will it take to fill the cistern when all three pipes are running?

6. A man can do a piece of work in 16 days, another in 18 days, and a third in 15 days. How many days will it require all to do it when working together?

7. *A* can do a piece of work in *a* days, *B* can do it in *b* days, *C* in *c* days, and *D* in *d* days. How long will it require all to do it when working together?

PROBLEMS ON THERMOMETER READINGS

There are two kinds of thermometers in use in this country, called the Fahrenheit and Centigrade, the former for common purposes, and the latter for scientific records and investigations. Hence, it frequently becomes necessary to translate readings from one kind to the other.



The freezing and boiling points are two fixed temperatures by means of which the computations are made. On the Centigrade these are marked 0° and 100° respectively and on the Fahrenheit they are marked 32° and 212° respectively. See the cut. Hence between the two fixed points there are 100 degrees Centigrade and 180 degrees Fahrenheit.

That is, 100 degree spaces on the Centigrade correspond to 180 degree spaces on the Fahrenheit.

Hence a change of 1° *Centigrade corresponds to* a change of $\frac{9}{5}$° *Fahrenheit*, and a change of 1° *Fahrenheit corresponds to* a change of $\frac{5}{9}$° *Centigrade*.

All problems comparing the two thermometers are solved by reference to these fundamental relations.

8. If the temperature falls 15 degrees Centigrade, how many degrees Fahrenheit does it fall?

9. If the temperature rises 18 degrees Fahrenheit, how many degrees Centigrade does it rise?

10. Translate $+25°$ Centigrade into Fahrenheit reading.

25° Centigrade equals $\frac{9}{5} \cdot 25° = 45°$ Fahrenheit.

45° above the freezing point $= 45° + 32°$ above 0° Fahrenheit. Hence, calling the Fahrenheit reading F, we have $F = 32 + \frac{9}{5} \cdot 25$.

11. Translate $+14°$ Centigrade into Fahrenheit reading.

Reasoning as before, $F = 32 + \frac{9}{5} \cdot 14$.

From the two preceding problems we have the formula

$$\mathbf{F = 32 + \tfrac{9}{5}C}. \qquad (1)$$

Translate this into words, understanding that F and C stand for readings on the respective thermometers.

12. Solve the above equation for C in terms of F and find

$$\mathbf{C = \tfrac{5}{9}(F - 32)}. \qquad (2)$$

13. By substituting in the equation $F = 32 + \frac{9}{5}C$, find the Fahrenheit reading when the Centigrade is 60°; 100°; $-40°$ or the freezing point of mercury.

14. In like manner find the Centigrade reading when the Fahrenheit is $-40°$; $-20°$; 98° or blood heat; 212° or the boiling point.

15. What is the temperature Centigrade when the sum of the Centigrade and Fahrenheit readings is 102°?

16. What is the temperature Fahrenheit when the sum of the Centigrade and Fahrenheit readings is zero?

17. What is the temperature Centigrade when the sum of the Centigrade and Fahrenheit readings is 140°?

18. What is the temperature in each reading when the Fahrenheit is 50° higher than the Centigrade?

CHAPTER VII

EQUATIONS OF THE FIRST DEGREE IN TWO OR MORE UNKNOWNS

105. The degree of an integral equation in two or more letters is the sum of the exponents of those letters in that one of its terms in which this sum is greatest.

E.g. $y = 2x + 1$ is of the *first* degree in x and y. $y^2 = 2x + y$ and $y = 2xy + 3$ are each of the *second* degree in x and y.

106. An Indeterminate Equation. If a single equation contains two unknowns, an unlimited number of pairs of values may be found which satisfy the equation.

Thus, in the equation $y = 2x + 1$, if $x = 1$, $y = 2 \cdot 1 + 1 = 3$; and if $x = -2$, $y = 2(-2) + 1 = -3$.

Hence, $x = 1$, $y = 3$; and $x = -2$, $y = -3$ are pairs of numbers which satisfy this equation.

For this reason a single equation in two unknowns is called *an indeterminate equation*, and the unknowns are sometimes called *variables.* A solution of such an equation is any pair of numbers which satisfy it.

Solutions of the equation $y = 2x + 1$ may be found by assigning any value to x and computing the corresponding value of y. Thus, we may take

$$x = 0, 1, 2, 3, 4, 5, 6, 7, \text{ etc., and find}$$
$$y = 1, 3, 5, 7, 9, 11, 13, 15, \text{ etc.}$$

or we may take $x = -1, -2, -3, -4, -5, -6, -7$, etc., and find

$$y = -1, -3, -5, -7, -9, -11, -13, \text{ etc.}$$

107. Two Indeterminate Equations. In the case of two equations, each in two unknowns, there are in general *only a limited number of values* which satisfy both equations.

E.g. The equations $y = 2x + 1$ and $y = 5x - 2$ are both satisfied by $x = 1$, $y = 3$, and by *no other pair* of values of x and y.

INDEPENDENT AND DEPENDENT EQUATIONS

108. Simultaneous Equations. Two equations in two unknowns which are satisfied by the same pair of values of the unknowns are called *simultaneous, or consistent, equations.*

E.g. $x + y = 5$ and $x - y = 3$ are simultaneous, because both are satisfied by $x = 4$, $y = 1$.

109. Contradictory Equations. Two equations in two unknowns, which cannot be satisfied by the same pair of values of the unknowns, are called *contradictory, or inconsistent, equations.*

E.g. $x + y = 5$ and $x + y = 2$ are contradictory, since no pair of values of x and y can make their sum both 5 and 2 at the same time.

110. Dependent Equations. Two equations in two unknowns are said to be *dependent* if one can be derived from the other by any of the processes used in deriving equivalent equations (§ 97). In this case *every pair of values which satisfies one will also satisfy the other.*

E.g. $x + y = 5$ and $2x + 2y = 10$ are dependent, since the second may be derived from the first by multiplying both members by 2.

111. Independent Equations. Two equations in two unknowns are *independent* if neither can be derived from the other.

E.g. $x + y = 5$ and $x - y = 3$ are independent equations.

If two equations of the first degree in x and y are not only simultaneous, but also independent, they have one and only one pair of values of x and y which together satisfy both.

E.g. $y = 2x + 1$ and $y = 5x - 2$ are simultaneous and independent, and they are satisfied by $x = 1$, $y = 3$, and by no other pair of values.

112. Solution by Elimination. The solution of a pair of simultaneous and independent equations of the first degree in two unknowns may be obtained by a process called *elimination.*

This process consists in combining the equations in such a way as to derive a single equation containing only one of the unknowns.

113. Elimination by Addition or Subtraction. One of the unknowns may be eliminated by *addition or subtraction* when the coefficients of that unknown are equal numerically, or are made equal, in the two equations.

Example. Solve
$$\begin{cases} 2x + 3y = 7, & (1) \\ 5x - 2y = 8. & (2) \end{cases}$$

Solution. In this case the coefficients of neither x nor y are numerically equal in the two equations, but if we multiply equation (1) by 2, and equation (2) by 3, the coefficients of y will be made so.

Then we have
$$\begin{cases} 4x + 6y = 14, & (3) \\ 15x - 6y = 24. & (4) \end{cases}$$

If now we add equations (3) and (4), $6y$ and $-6y$ will cancel, and we have the equation
$$19x = 38, \qquad (5)$$
which contains only the variable x.

Dividing equation (5) by 19, we get $x = 2$.

Substituting $x = 2$ in equation (1), we find
$$2 \cdot 2 + 3y = 7,$$
from which $3y = 3$, and $y = 1$.

Hence, $\begin{cases} x = 2 \\ y = 1 \end{cases}$ is the solution of the pair of equations (1) and (2).

114. Elimination by Substitution. An equation of the first degree in two unknowns may always be solved for one of these unknowns in terms of the other. If the value of the unknown thus found in one equation be substituted in the other, then an equation is derived containing only one unknown. This is called *elimination by substitution.*

Example. Solve
$$\begin{cases} 2x + 3y = 7, & (1) \\ 5x - 2y = 8. & (2) \end{cases}$$

Solution. Solving equation (1) for y in terms of x, we get
$$y = \frac{7 - 2x}{3}. \qquad (3)$$

Substituting this value of y in equation (2),
$$5x - 2 \cdot \frac{7 - 2x}{3} = 8. \qquad (4)$$

Solving, $x = 2.$

Substituting $x = 2$ in equation (1), $y = 1.$

115. Elimination by Comparison. A third method of elimination consists in expressing the same unknown in terms of the other in each equation and equating these two expressions to each other. This is called *elimination by comparison.*

Example. Solve $\begin{cases} 3y + x = 14, & (1) \\ 2y - 5x = -19. & (2) \end{cases}$

Solving (1) for x, $x = 14 - 3y.$ (3)

Solving (2) for x, $x = \frac{19 + 2y}{5}.$ (4)

From (3) and (4), by axiom 5, § 96,

$$14 - 3y = \frac{19 + 2y}{5}. \qquad (5)$$

Solving (5), $y = 3.$
Substituting in (1), $x = 5.$
Check by substituting $x = 5$, $y = 3$ in (2).

In applying any method of elimination it is desirable first to reduce each equation to the standard form:

$$ax + by = c.$$

ORAL EXERCISES

Solve the following pairs of equations by addition or subtraction:

1. $\begin{cases} x + y = 12, \\ x - y = 4. \end{cases}$

2. $\begin{cases} 2x + y = 16, \\ x - y = 5. \end{cases}$

3. $\begin{cases} 2x + 3y = 13, \\ -2x + 5y = 11. \end{cases}$

4. $\begin{cases} x - 2y = 1, \\ 3x + 2y = 19. \end{cases}$

5. $\begin{cases} 5x + 3y = 11, \\ 2x - 3y = -1 \end{cases}$

6. $\begin{cases} 3x - 2y = 25, \\ 5x - 2y = 39. \end{cases}$

7. $\begin{cases} -2x + 5y = -3, \\ 2x - y = -1. \end{cases}$

8. $\begin{cases} 3x + 5y = 11, \\ 7x + 5y = 19. \end{cases}$

9. $\begin{cases} 3x - 4y = 1, \\ 5x + 4y = 23. \end{cases}$

10. $\begin{cases} 4x - 3y = 8, \\ 2x - 3y = -2. \end{cases}$

WRITTEN EXERCISES

Solve the following pairs of equations by any one of the processes of elimination:

1. $\begin{cases} 3x + 2y = 118, \\ x + 5y = 191. \end{cases}$ 2. $\begin{cases} 5x - 8\frac{1}{2} = 7y - 44, \\ 2x = y + \frac{5}{7}. \end{cases}$

3. $\begin{cases} 6x - 3y = 7, \\ 2x - 2y = 3. \end{cases}$ 4. $\begin{cases} 3x + 7y - 341 = 7\frac{1}{2}y + 43\frac{1}{2}x, \\ 2\frac{1}{2}x + \frac{1}{2}y = 1. \end{cases}$

5. $\begin{cases} 5x - 11y - 2 = 4x, \\ 5x - 2y = 63. \end{cases}$ 6. $\begin{cases} 3y + 40 = 2x + 14, \\ 9y - 347 = 5x - 420. \end{cases}$

7. $\begin{cases} 5y - 3x + 8 = 4y + 2x + 7, \\ 4x - 2y = 3y + 2. \end{cases}$ 8. $\begin{cases} 6y - 5x = 5x + 14, \\ 3y - 2x - 6 = 5 + x. \end{cases}$

9. $\begin{cases} (x+5)(y+7) = (x+1)(y-9) + 112, \\ 2x + 10 = 3y + 1. \end{cases}$ 10. $\begin{cases} 73 - 7y = 5x, \\ 2y - 3x = 12. \end{cases}$

11. $\begin{cases} ax = by, \\ x + y = c. \end{cases}$ 13. $\begin{cases} x + y = a, \\ x - y = b. \end{cases}$ 15. $\begin{cases} \dfrac{3}{x} - \dfrac{5}{y} = 6, \\ \dfrac{2}{x} + \dfrac{3}{y} = 2. \end{cases}$

12. $\begin{cases} \dfrac{x}{a} = \dfrac{y}{b}, \\ x + y = s. \end{cases}$ 14. $\begin{cases} ax + by = c, \\ fx + gy = h. \end{cases}$ 16. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = c, \\ \dfrac{f}{x} + \dfrac{g}{y} = h. \end{cases}$

Note.—In examples 15 and 16, consider $\frac{1}{x}$ and $\frac{1}{y}$ as the unknowns.

17. $\begin{cases} a(x + y) - b(x - y) = 2a, \\ a(x - y) - b(x + y) = 2b. \end{cases}$

18. $\begin{cases} (k + 1)x + (k - 2)y = 3a, \\ (k + 3)x + (k - 4)y = 7a. \end{cases}$

19. $\begin{cases} 2ax + 2by = 4a^2 + b^2, \\ x - 2y = 2a - b. \end{cases}$

20. $\begin{cases} (a + b)x - (a - b)y = 4ab, \\ (a - b)x + (a + b)y = 2a^2 - 2b^2. \end{cases}$

21. $\begin{cases} \frac{1}{2}(a-b)-\frac{1}{5}(a-3b)=b-3, \\ \frac{3}{4}(a-b)+\frac{5}{6}(a+b)=18. \end{cases}$

22. $\begin{cases} a(x+y)+b(x-y)=2, \\ a^2(x+y)-b^2(x-y)=a-b. \end{cases}$

23. $\begin{cases} 7(x-5)=3-\dfrac{y}{2}-x, \\ \frac{1}{4}(x-y)+\frac{1}{2}y-\frac{5}{8}(x-1)=-1. \end{cases}$

24. $\begin{cases} mx+ny=m^3+2m^2n+n^3, \\ nx+my=m^3+2mn^2+n^3. \end{cases}$

25. $\begin{cases} (m+n)x-(m-n)y=2lm, \\ (m+l)x-(m-l)y=2mn. \end{cases}$

26. $\begin{cases} (a-b)x+(a+b)y=2a, \\ (a+b)x+(a-b)y=2b. \end{cases}$

27. $\begin{cases} (h+k)x+(h-k)y=2(h^2+k^2), \\ (h-k)x+(h+k)y=2(h^2-k^2). \end{cases}$

28. $\begin{cases} (a-b)x+y(a^2+b^2)=(a+b)^2+a+b-2ab, \\ (b-a)x+y(a^2+b^2)=a+b-a^2-b^2. \end{cases}$

In examples 29 to 34 solve without clearing of fractions.

29. $\begin{cases} \dfrac{5}{t}-\dfrac{6}{v}=2, \\ \dfrac{17}{v}+\dfrac{4}{t}=67. \end{cases}$

30. $\begin{cases} \dfrac{3}{x}+\dfrac{1}{y}=21, \\ \dfrac{7}{x}-\dfrac{9}{y}=-19. \end{cases}$

31. $\begin{cases} \dfrac{7}{a}-\dfrac{1}{b}=12\frac{1}{2}, \\ \dfrac{3}{a}+\dfrac{12}{b}=24. \end{cases}$

32. $\begin{cases} \dfrac{3}{x}-\dfrac{2}{y}=-4, \\ \dfrac{6}{x}+\dfrac{11}{y}=52. \end{cases}$

33. $\begin{cases} \dfrac{12}{x}-\dfrac{10}{y}=7, \\ \dfrac{9}{x}+\dfrac{2}{y}=10. \end{cases}$

34. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=1, \\ \dfrac{c}{x}+\dfrac{d}{y}=1. \end{cases}$

SYSTEMS OF EQUATIONS IN MORE THAN TWO UNKNOWNS

116. Systems of Equations. Two or more equations involving the same unknowns form, when taken together, a *system of equations.* A system containing as many unknowns as equations may be *simultaneous* or *contradictory, independent* or *dependent,* in the same sense, as explained in §§ 108–111 for two equations in two unknowns.

117. Indeterminate Equations. If a *single* equation of the first degree in three or more variables is given, there is no limit to the number of sets of values which satisfy it.

E.g. $3x + 2y + 4z = 24$ is satisfied by $x = 1$, $y = 3$, $z = 3\frac{3}{4}$; $x = 2$, $y = 2$, $z = 3\frac{1}{2}$; $x = 0$, $y = 0$, $z = 6$; etc.

If *two* equations of the first degree in three or more variables are given, they have in general an unlimited number of solutions.

E.g. $3x + 2y + 4z = 24$ and $x + y + z = 6$ are both satisfied by $x = 2$, $y = -1$, $z = 5$; $x = 3$, $y = -1\frac{1}{2}$, $z = 4\frac{1}{2}$; etc.

But if a system of equations of the first degree contains *as many equations as variables,* it has in general one and only one set of values which together satisfy all the equations.

E.g. The system
$$\begin{cases} x + y + z = 6, & (1) \\ 3x - y + 2z = 7, & (2) \\ 2x + 3y - z = 5 & (3) \end{cases}$$
is satisfied by $x = 1$, $y = 2$, $z = 3$, and by no other set of values.

This is the case when the system is *independent* and *simultaneous.*

ORAL EXERCISES

1. If equations (1) and (2) in the above example are added, will the resulting equation contain y?

2. If equation (1) is multiplied by 2 and equation (2) subtracted from it, will the resulting equation contain y? Will it contain z?

3. How would you eliminate z from equations (2) and (3)?

4. How would you eliminate y from equations (2) and (3)?

SOLUTION OF EQUATIONS IN THREE UNKNOWNS

118. Solution of a System of three Equations. An independent and simultaneous system of equations of the first degree in three unknowns may be solved as follows:

From two of the equations, say the 1st and 2d, eliminate one of the unknowns, obtaining one equation in the remaining two unknowns.

From the 1st and 3d equations, or from the 2d and 3d, eliminate the same unknown, obtaining a second equation in the remaining two unknowns.

Solve as usual the two equations in two unknowns thus found. Substitute the values of these two unknowns in one of the given equations, and thus find the value of the third unknown.

The process of elimination by addition or subtraction is usually most convenient.

Example. Solve the system:
$$\begin{cases} x + y + z = 9, & (1) \\ 2x + 3y + z = 17, & (2) \\ x + 2y + 2z = 16. & (3) \end{cases}$$

Solution. If we subtract equation (1) from equation (3), we get an equation containing only y and z,
$$y + z = 7. \qquad (4)$$

If we multiply equation (3) by 2, and then subtract equation (2) from it, we get a second equation containing only y and z,
$$y + 3z = 15. \qquad (5)$$

Hence, we now have the system
$$\begin{cases} y + z = 7, \\ y + 3z = 15, \end{cases}$$
from which we solve, as in § 113, obtaining the values of y and z,
$$y = 3, \ z = 4.$$

Substituting these values of y and z in equation (1), we find
$$x = 2.$$

Hence, the solution of the system is
$$\begin{cases} x = 2, \\ y = 3, \\ z = 4. \end{cases}$$

We may also find the solution by first eliminating y, using (1) and (2), and then using (2) and (3), getting two equations in x and z, from which the values of x and z can be found.

Evidently z could have been eliminated first, using equations (1) and (2) and then (1) and (3), giving a new set of two equations in y and x. Let the student find the solution in this manner.

EXERCISES

Solve each of the following systems of equations:

1. $\begin{cases} 2x + 5y - 7z = 9, \\ 5x - y + 3z = 16, \\ 7x + 6y + z = 34. \end{cases}$

2. $\begin{cases} a + b + c = 9, \\ 8a + 4b + 2c = 36, \\ 27a + 9b + 3c = 93. \end{cases}$

3. $\begin{cases} 18l - 7m - 5n = 161, \\ 4\frac{2}{5}m - \frac{2}{3}l + n = 18, \\ 3\frac{1}{2}n + 2m + \frac{3}{4}l = 33. \end{cases}$

4. $\begin{cases} \dfrac{a}{3} + \dfrac{b}{6} + \dfrac{c}{9} = -2, \\ \dfrac{a}{6} + \dfrac{b}{9} + \dfrac{c}{12} = -4, \\ \dfrac{a}{9} + \dfrac{b}{12} + \dfrac{c}{15} = -4. \end{cases}$

5. $\begin{cases} 8z - 3y + x = -2, \\ 3x - 5y - 6z = -46, \\ y + 5x - 4z = -18. \end{cases}$

6.* $\begin{cases} \dfrac{3}{a} = \dfrac{2}{b}, \\ \dfrac{2}{a} + \dfrac{5}{b} - \dfrac{4}{c} = 17, \\ \dfrac{7}{a} - \dfrac{3}{b} + \dfrac{6}{c} = 8. \end{cases}$

7. $\begin{cases} x + 2y - 3z = -3, \\ 2x - 3y + z = 8, \\ 5x - 4y - 7z = -5. \end{cases}$

8. $\begin{cases} 2x + 3y - 7z = 19, \\ 5x + 8y + 11z = 24, \\ 7x + 11y + 4z = 43. \end{cases}$

Show that this system is not independent.

9. $\begin{cases} x + y = 16, \\ z + x = 22, \\ y + z = 28. \end{cases}$

10. $\begin{cases} x + 2y = 26, \\ 3x + 4z = 56, \\ 5y + 6z = 65. \end{cases}$

11. $\begin{cases} a + b + c = 5, \\ 3a - 5b + 7c = 79, \\ 9a - 11b = 91. \end{cases}$

12. $\begin{cases} l + m + n = 29\frac{1}{4}, \\ l + m - n = 18\frac{1}{4}, \\ l - m + n = 13\frac{3}{4}. \end{cases}$

13. $\begin{cases} l + m + n = a, \\ l + m - n = b, \\ l - m + n = c. \end{cases}$

14. $\begin{cases} ax + by = p, \\ cy + dz = q, \\ ex + fz = r. \end{cases}$

* Use $\dfrac{1}{a}$, $\dfrac{1}{b}$, and $\dfrac{1}{c}$ as the unknowns.

15. $\begin{cases} \dfrac{1}{a}+\dfrac{1}{b}=4, \\ \dfrac{1}{a}+\dfrac{1}{c}=3, \\ \dfrac{1}{b}+\dfrac{1}{c}=2. \end{cases}$

16. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a, \\ \dfrac{1}{x}+\dfrac{1}{z}=b, \\ \dfrac{1}{y}+\dfrac{1}{z}=c. \end{cases}$

17. $\begin{cases} u+v+x+y=10, \\ 2u-3v+4x-5y=-12, \\ 3u+4v-5x+6y=20, \\ 4u+5v+6x-7y=4. \end{cases}$

18. Make a rule for solving a system of four or more linear equations in as many unknowns as equations.

PROBLEMS INVOLVING TWO OR MORE UNKNOWNS

1. A man invests a certain amount of money at 4 % interest and another amount at 5 %, thereby obtaining an annual income of $3100. If the first amount had been invested at 5 % and the second at 4 %, the income would have been $3200. Find each investment.

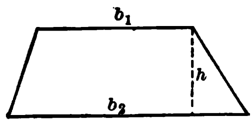
2. The relation between the readings of the Centigrade and the Fahrenheit thermometers is given on page 68 by the equation $F = 32 + \frac{9}{5}C$. The Fahrenheit reading at the melting temperature of tungsten is 2432 degrees higher than the Centigrade. Find the melting temperature in each scale.

3. Going with a current a steamer makes 19 miles per hour, while going against a current $\frac{3}{4}$ as strong it makes 5 miles per hour. Find the speed of each current and the boat.

4. There is a number consisting of 3 digits whose sum is 11. If the digits are written in reverse order, the resulting number is 594 less than the original number. Three times the tens' digit is one more than the sum of the hundreds' and the units' digits.

5. A certain kind of wine contains 20 % alcohol and another kind contains 28 %. How many gallons of each must be used to form 50 gallons of a mixture containing 21.6 % alcohol?

6. The area of a certain trapezoid of altitude 8 is 68. If 4 is added to the lower base and the upper base is doubled, the area is 108. Find both bases.



A trapezoid is a four-sided figure whose upper base, b_1, and lower base, b_2, are parallel, but the other two sides are not. If h is the perpendicular distance between the bases, then the area is $a = \frac{h}{2}(b_1 + b_2)$.

7. The Centigrade reading at the boiling point of alcohol is 96° lower than the Fahrenheit reading. Find both the Centigrade and the Fahrenheit readings at this temperature.

Use C and F as the unknowns. Then one of the equations is the formula connecting Fahrenheit and Centigrade readings obtained on page 68 and the other is $C + 96 = F$.

8. The Fahrenheit reading at the temperature of liquid air is 128 degrees lower than the Centigrade reading. Find both the Centigrade and the Fahrenheit readings at this temperature.

9. The Centigrade reading at the melting point of silver is 796° lower than the Fahrenheit reading. Find both Centigrade and Fahrenheit readings at this temperature.

10. The Fahrenheit reading at the melting point of gold is 992° higher than the Centigrade reading. Find both Centigrade and Fahrenheit readings at this temperature.

11. The upper base of a trapezoid is 6 and its area is 168. If $\frac{1}{8}$ the lower base is added to the upper, the area is 210. Find the altitude and the lower base.

12. A and B can do a piece of work in 18 days, B and C in 24 days, and C and A in 36 days. How long will it require each man to do it working alone, and how long will it require all working together?

13. A and B can do a piece of work in m days, B and C in n days, and C and A in p days. How long will it require each to do it working alone?

14. A beam resting on a fulcrum balances when it carries weights of 100 and 130 pounds at its respective ends. The beam will also balance if it carries weights of 80 and 110 pounds respectively, 2 feet from the ends, provided the fulcrum remains in the same place. Find the distance from the fulcrum to the ends of the beam.

15. A beam carries three weights, A, B, C. A balance is obtained when A is 12 feet from the fulcrum, B is 8 feet from the fulcrum on the same side as A, and C is 20 feet from the fulcrum on the side opposite A. It also balances when the distance of A is 8 feet, B is 10 feet, and C is 18 feet. Find the weights B and C if A is 50 pounds.

16. At 0° Centigrade sound travels 1115 feet per second with the wind on a certain day, and 1065 feet per second against the wind. Find the velocity of sound in calm weather, and the velocity of the wind on this occasion.

17. If the velocities of sound in air, brass, and iron at 0° Centigrade are x, y, z meters per second respectively, then $3x + 2y - z = 2505$, $5x - 2y + z = 151$, and $x + y + z = 8777$. Find the velocity in each.

18. If x, y, z are the Centigrade readings at the temperatures which liquefy hydrogen, nitrogen, and oxygen respectively, then $3x - 8y + 2z = 440$, $-8x + 2y + 4z = 903$, and $x + 4y - 6z = 60$. Find each temperature in both Centigrade and Fahrenheit readings.

19. If x, y, z are the Centigrade readings at the freezing temperatures of hydrogen, nitrogen, and oxygen respectively, then we have $x + y - 3z = 199$, $2x - 5y + z = 328$, and $-4x + 2y + 2z = 156$. Find each temperature.

20. If x, y, z are, respectively, the melting point of carbon, the temperature of the hydrogen flame in air, and the temperature of this flame in pure oxygen, then $10x + 2y + z = 41892$, $15x + y + 2z = 60212$, and $7x + y + z = 29368$. Find each.

21. Two boys carry a 120-pound weight by means of a pole, at a certain point on which the weight is hung. One boy holds the pole 5 ft. from the weight and the other 3 ft. from it. What part of the weight does each boy lift?

Suggestion. Let x and y be the required amounts, then $5x$ is the leverage of the first boy and $3y$ that of the second, and these must be equal as in the case of the teeter. Hence, we have

$$5x = 3y, \text{ and } x + y = 120.$$

22. If, in problem 21, the boys lift P_1 and P_2 pounds respectively, at distances d_1 and d_2, and w is the weight lifted, then

$$P_1d_1 = P_2d_2. \qquad (1)$$

$$P_1 + P_2 = w. \qquad (2)$$

Solve (1) and (2), (a) when P_1 and P_2 are unknown, (b) when P_1 and w are unknown, (c) when P_1 and d_2 are unknown.

23. A weight of 540 pounds is carried on a pole by two men at distances of 4 and 5 feet respectively, from the weight. How much does each lift?

24. There is a number consisting of three digits such that the sum of the hundreds' and units' digits is 7. If the order of the digits is reversed the number is increased by 297 and if the tens' and hundreds' digits are interchanged the number is increased by 450. Find the number.

CHAPTER VIII

SYSTEMS OF EQUATIONS SOLVED BY DETERMINANTS

SYSTEMS OF TWO EQUATIONS

119. Solution of Equations with Numerical Coefficients. We now proceed to a more general study of the solution of a pair of first-degree equations in two unknowns.

Example 1. Solve

$$\begin{cases} 2x+3y=4, & (1) \\ 5x+6y=7. & (2) \end{cases}$$

Solution. Multiplying (1) by 5 and (2) by 2,

$$\begin{cases} 5\cdot 2x+5\cdot 3y=5\cdot 4, & (3) \\ 2\cdot 5x+2\cdot 6y=2\cdot 7. & (4) \end{cases}$$

Subtracting (3) from (4), $(2\cdot 6-5\cdot 3)y=2\cdot 7-5\cdot 4.$ (5)

Solving for y, $y=\dfrac{2\cdot 7-5\cdot 4}{2\cdot 6-5\cdot 3}=\dfrac{-6}{-3}=2.$ (6)

In like manner, solve for x by multiplying (1) by 6 and (2) by 3 and subtracting. We then have

$$=\frac{4\cdot 6-7\cdot 3}{2\cdot 6-5\cdot 3}=\frac{3}{-3}=-1. \quad (7)$$

The expressions, $x=\dfrac{4\cdot 6-7\cdot 3}{2\cdot 6-5\cdot 3}$ and $y=\dfrac{2\cdot 7-5\cdot 4}{2\cdot 6-5\cdot 3}$ show exactly how each coefficient in the original equations enters into the solution, while $x=-1$, $y=2$ give no information on this point.

Example 2. In this manner, solving,

$$\begin{cases} 7x+9y=71, \\ 2x+3y=48, \end{cases}$$

we find $x=\dfrac{71\cdot 3-48\cdot 9}{7\cdot 3-2\cdot 9}$ and $y=\dfrac{7\cdot 48-2\cdot 71}{7\cdot 3-2\cdot 9}.$

In Example 2, the various coefficients are found to occupy the same relative positions in the expressions for x and y as do the corresponding coefficients in Example 1.

120. Solution of General Equations. A convenient rule for reading directly the values of the unknowns in such pairs of equations as the above may be made from the solution of the following *general equations:*

Example 3. Solve
$$\begin{cases} a_1x + b_1y = c_1, & (1) \\ a_2x + b_2y = c_2. & (2) \end{cases}$$

Solution. To eliminate y, multiply equation (1) by b_2 and equation (2) by b_1, getting
$$\begin{cases} a_1b_2x + b_1b_2y = c_1b_2, & (3) \\ a_2b_1x + b_1b_2y = c_2b_1. & (4) \end{cases}$$

Subtracting equation (4) from (3),
$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1.$$

Dividing by the coefficient of x,
$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}. \qquad (5)$$

Again, to eliminate x multiply equation (1) by a_2, and equation (2) by a_1, and subtract (2) from (1), getting
$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \qquad (6)$$

121. Rule for Reading the Results. To remember these results, notice that the coefficients of x and y in the given equation stand in the form of a square, thus

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

and that the denominator in the expressions for both x and y is the diagonal a_1b_2 of this square, minus the diagonal a_2b_1.

The numerator in the expression for x is formed by replacing the a's in this square by the c's, thus,

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},$$

and then reading the diagonals as before obtaining $c_1b_2 - c_2b_1$.

The numerator for y is formed by replacing the b's by the c's, thus

$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},$$

and then reading the diagonals, obtaining $a_1c_2 - a_2c_1$.

122. Determinants of the Second Order. A square array like $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ with two numbers on each side, is called a *determinant of the second order* and its value is given by the equation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \qquad (A)$$

Thus, $\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}$ means $4 \cdot 1 - 3 \cdot 2 = -2.$

123. Determinant Formulas. Since any pair of equations of the first degree in two unknowns may be reduced to the *standard form* as given in Example 3, page 83, it follows that the values of x and y there obtained constitute a formula for the solution of any pair of such equations.

Example 4. Solve by determinants:

$$\begin{cases} 5x = 8 + 3y, & (1) \\ 7y = 19 - 2x. & (2) \end{cases}$$

Solution. Putting these equations in the standard form,

$$\begin{cases} 5x - 3y = 8, & (3) \\ 2x + 7y = 19. & (4) \end{cases}$$

The determinant of the coefficients is

$$\begin{vmatrix} 5 & -3 \\ 2 & 7 \end{vmatrix} = 5 \cdot 7 - (-3) \cdot 2 = 35 + 6 = 41.$$

This determinant with the coefficients of x replaced by the known terms is

$$\begin{vmatrix} 8 & -3 \\ 19 & 7 \end{vmatrix} = 8 \cdot 7 - (-3) \cdot 19 = 56 + 57 = 113.$$

This determinant with the coefficients of y replaced by the known terms is

$$\begin{vmatrix} 5 & 8 \\ 2 & 19 \end{vmatrix} = 5 \cdot 19 - 2 \cdot 8 = 95 - 16 = 79.$$

Hence, the solution is

$$x = \frac{\begin{vmatrix} 8 & -3 \\ 19 & 7 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 2 & 7 \end{vmatrix}} = \frac{113}{41}, \text{ and } y = \frac{\begin{vmatrix} 5 & 8 \\ 2 & 19 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 2 & 7 \end{vmatrix}} = \frac{79}{41}.$$

124. Rule for Solving two Equations of the first Degree by Determinants. These examples lead to the following rule:

(1) *Reduce the equations to the standard form*

$$a_1x + b_1y = c_1,$$

and

$$a_2x + b_2y = c_2$$

(2) *Form the determinant of the coefficients*

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

(3) *Find the determinant obtained by replacing the a's by the c's, and also the one obtained by replacing the b's by c's*, i.e.,

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

(4) *Write the values of x and y in the form*

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

(5) *Evaluate these determinants thus:*

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

In practice the values of x and y are written down at once as in (4), omitting step (3).

The solution by means of determinants is especially advantageous in case of systems of literal equations.

ORAL EXERCISES

In each of the following give the determinant of the coefficients, the x-numerator, and the y-numerator:

1. $\begin{cases} 3x + 2y = 4, \\ 5x + 3y = 9. \end{cases}$

2. $\begin{cases} 7x + y = 7, \\ 3x + 2y = 14. \end{cases}$

3. $\begin{cases} 4x - y = 12, \\ x - 5y = 3. \end{cases}$

4. $\begin{cases} 2x + 5y = 15, \\ 3x - 4y = 7. \end{cases}$

WRITTEN EXERCISES

Solve the following system by means of determinants.

1. $\begin{cases} 3x + 4y = 10, \\ 4x + y = 9. \end{cases}$

2. $\begin{cases} 4x - 5y = -26, \\ 2x - 3y = -14. \end{cases}$

3. $\begin{cases} 6y - 17 = -5x, \\ 6x - 16 = -5y. \end{cases}$

4. $\begin{cases} \frac{1}{4}(x-3) = -\frac{1}{3}(y-2) + \frac{1}{2}x, \\ \frac{1}{2}(y-1) = x - \frac{1}{3}(x-2). \end{cases}$

5. $\begin{cases} 2x - y = 53, \\ 19x + 17y = 0. \end{cases}$

6. $\begin{cases} ax - by = 0, \\ x - y = c. \end{cases}$

7. $\begin{cases} mx + ny = p, \\ rx + sy = t. \end{cases}$

8. $\begin{cases} x(a-b) + y(a+b) = 2a, \\ x(a+b) - y(a-b) = 2b. \end{cases}$

9. $\begin{cases} (k+1)x + (k-2)y = 3a, \\ (k+3)x + (k-4)y = 7a. \end{cases}$

10. $\begin{cases} 2ax + 2by = 4a^2 + b^2, \\ x - 2y = 2a - b. \end{cases}$

11. $\begin{cases} (a+b)x - (a-b)y = 4ab, \\ (a-b)x + (a+b)y = 2a^2 - 2b^2. \end{cases}$

12. $\begin{cases} mx + ny = m^3 + 2m^2n + n^3, \\ nx + my = m^3 + 2mn^2 + n^3. \end{cases}$

13. $\begin{cases} (m+n)x - (m-n)y = 2lm, \\ (m+l)x - (m-l)y = 2mn. \end{cases}$

14. $\begin{cases} (a-b)x + (a+b)y = 2a, \\ (a-b)x - (a+b)y = 2b. \end{cases}$

15. $\begin{cases} (h+k)x + (h-k)y = 2(h^2 + k^2), \\ (h-k)x + (h+k)y = 2(h^2 - k^2). \end{cases}$

16. $\begin{cases} (a-b)x + y(a^2 + b^2) = a, \\ (b-a)x + y(a^2 + b^2) = b. \end{cases}$

17. $\begin{cases} a(x+y) + b(x-y) = 2, \\ a^2(x+y) - b^2(x-y) = a - b. \end{cases}$

18. $\begin{cases} 7(x-5) = 3 - \dfrac{y}{2} - x, \\ \frac{1}{4}(x-y) + \frac{1}{2}y - \frac{5}{8}(x-1) = -1. \end{cases}$

19. $\begin{cases} \frac{1}{2}(a-b) - \frac{1}{5}(a-3b) = b - 3, \\ \frac{3}{4}(a-b) + \frac{5}{6}(a+b) = 18. \end{cases}$

SYSTEMS OF THREE EQUATIONS

125. Solution by Determinants. Systems of three equations of the first degree in three unknowns may be solved by determinants in a manner similar to the foregoing solution of systems of two equations.

Example. Solve the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1, & (1) \\ a_2x + b_2y + c_2z = d_2, & (2) \\ a_3x + b_3y + c_3z = d_3. & (3) \end{cases}$$

Solution. Multiply equation (1) by c_2 and equation (2) by c_1, and subtract (2) from (1), thus eliminating z from these equations. Then multiply equation (1) by c_3, and equation (3) by c_1, and subtract (3) from (1), thus eliminating z from these equations.

We then have the system of two equations in x and y.

$$(a_1c_2 - a_2c_1)x + (b_1c_2 - b_2c_1)y = d_1c_2 - d_2c_1, \qquad (4)$$
$$(a_1c_3 - a_3c_1)x + (b_1c_3 - b_3c_1)y = d_1c_3 - d_3c_1. \qquad (5)$$

Solving this system of two equations by determinants, according to the rule (§ 124) we have for the value of x,

$$x = \frac{\begin{vmatrix} d_1c_2 - d_2c_1 & b_1c_2 - b_2c_1 \\ d_1c_3 - d_3c_1 & b_1c_3 - b_3c_1 \end{vmatrix}}{\begin{vmatrix} a_1c_2 - a_2c_1 & b_1c_2 - b_2c_1 \\ a_1c_3 - a_3c_1 & b_1c_3 - b_3c_1 \end{vmatrix}},$$

$$\text{or } x = \frac{(d_1c_2 - d_2c_1)(b_1c_3 - b_3c_1) - (d_1c_3 - d_3c_1)(b_1c_2 - b_2c_1)}{(a_1c_2 - a_2c_1)(b_1c_3 - b_3c_1) - (a_1c_3 - a_3c_1)(b_1c_2 - b_2c_1)}.$$

Performing the multiplications, we have for the value of x

$$\frac{\cancel{d_1b_1c_2c_3} - d_1b_3c_1c_2 - d_2b_1c_1c_3 + d_2b_3c_1^2 \cancel{- d_1b_1c_2c_3} + d_3b_1c_1c_2 + d_1b_2c_1c_3 - d_3b_2c_1^2}{\cancel{a_1b_1c_2c_3} - a_1b_3c_1c_2 - a_2b_1c_1c_3 + a_2b_3c_1^2 \cancel{- a_1b_1c_2c_3} + a_3b_1c_1c_2 + a_1b_2c_1c_3 - a_3b_2c_1^2}.$$

After cancelling $d_1b_1c_2c_3$ and $-d_1b_1c_2c_3$ in the numerator and $a_1b_1c_2c_3$ and $-a_1b_1c_2c_3$ in the denominator, we find that c_1 is a factor of every term in both numerator and denominator. Dividing this out, and writing the positive terms first, we have

$$x = \frac{d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}.$$

126. Expressing the Denominator as a Determinant. The denominator of the fraction in the above value of x contains the nine coefficients of x, y, and z in the given system of equations. If these be written in the form of a square array, thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

then the terms of the denominator may be read from this array as follows:

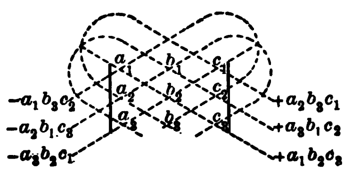
The Positive Terms. (1) *Take the diagonal* $a_1b_2c_3$.

(2) *Take the line* b_1c_2 *parallel to this diagonal, together with* a_3 *in the opposite corner, getting* $a_3b_1c_2$.

(3) *Take the line* a_2b_3 *also parallel to this diagonal, together with* c_1 *in the opposite corner, getting* $a_2b_3c_1$.

These are the positive terms: $a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1$.

The Negative Terms. (1) *Take the diagonal* $a_3b_2c_1$.



(2) *Take the line* a_2b_1 *parallel to this diagonal, together with* c_3 *in the opposite corner, getting* $a_2b_1c_3$.

(3) *Take the line* b_3c_2 *also parallel to this diagonal, together with* a_1 *in the opposite corner, getting* $a_1b_3c_2$.

These are the negative terms: $-a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$.

127. Determinant of the Third Order. A square array like the above is called a determinant of the third order. Its value is given by the equation:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

This is called the *determinant of the system* of equations in the example on page 87.

The scheme for reading it is shown in the above diagram.

128. Expressing the Numerators as Determinants. The numerator in the value of x on page 87 is exactly the same polynomial as the denominator with the a's replaced by the d's. Hence, this may also be written as a determinant of the third order by replacing the a's by the d's. Thus,

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2.$$

This is called the **x-determinant.** Similarly the **y-determinant** and **z-determinant** are formed by replacing the b's and c's in turn by the d's.

Hence we have,

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}.$$

If we solve the given system for y and z, in a manner similar to the above solution for x, we should find:

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{a_1d_2c_3 + a_3d_1c_2 + a_2d_3c_1 - a_3d_2c_1 - a_2d_1c_3 - a_1d_3c_2}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2},$$

and

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{a_1b_2d_3 + a_3b_1d_2 + a_2b_3d_1 - a_3b_2d_1 - a_2b_1d_3 - a_1b_3d_2}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}.$$

129. Determinant Formulas. The determinant values of x, y, and z may be used as formulas for solving any set of first degree equations in three unknowns. The formulas are directly applicable when the equations are reduced to the standard form (see § 125).

Example. By means of determinants solve the system:

$$\begin{aligned} y + \quad z &= 9 - x, \\ 2x + 3y &= 17 - z, \\ 2z + \quad x &= 16 - 2y. \end{aligned}$$

Solution. Putting the equations in the standard form, we have

$$\begin{aligned} x + \quad y + \quad z &= 9, \\ 2x + 3y + \quad z &= 17, \\ x + 2y + 2z &= 16. \end{aligned}$$

(1) The *determinant of the coefficients* is

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 6 + 1 + 4 - 3 - 4 - 2 = 2.$$

(2) The *x-*, *y-*, and *z-determinants* are formed by replacing the first, second, and third columns respectively, by the known terms, obtaining

$$\begin{vmatrix} 9 & 1 & 1 \\ 17 & 3 & 1 \\ 16 & 2 & 2 \end{vmatrix} = 54 + 16 + 34 - 48 - 34 - 18 = 4.$$

$$\begin{vmatrix} 1 & 9 & 1 \\ 2 & 17 & 1 \\ 1 & 16 & 2 \end{vmatrix} = 34 + 9 + 32 - 17 - 16 - 36 = 6.$$

$$\begin{vmatrix} 1 & 1 & 9 \\ 2 & 3 & 17 \\ 1 & 2 & 16 \end{vmatrix} = 48 + 17 + 36 - 27 - 34 - 32 = 8.$$

Hence,

$$x = \frac{\begin{vmatrix} 9 & 1 & 1 \\ 17 & 3 & 1 \\ 16 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}} = 2, \qquad y = \frac{\begin{vmatrix} 1 & 9 & 1 \\ 2 & 17 & 1 \\ 1 & 16 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}} = 3, \qquad z = \frac{\begin{vmatrix} 1 & 1 & 9 \\ 2 & 3 & 17 \\ 1 & 2 & 16 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}} = 4.$$

The complete solution is, then, $x = 2$, $y = 3$, $z = 4$.

130. Rule for Solving by Determinants. To solve a system of three equations of the first degree in three unknowns by means of determinants, we have the following rule:

(1) *Write the equations in the standard form,*

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

(2) *Form the determinant of the coefficients,* $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

(3) *Form the x-, y-, and z- determinants by replacing in the determinant of the coefficients the a's, b's, c's, respectively, by the d's,*

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

(4) *The solution is then*

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

(5) *Evaluate the determinants as shown on page 88.*

In practice, the values of x, y, z are written down at once as in (4), thus omitting steps (2) and (3).

It may seem to the student at first that this method of solution is longer than by ordinary elimination; but when facility is gained in evaluating determinants, it will be found to be a much more speedy and satisfactory method.

Determinants are used in solving first degree equations in four, five, or any number of unknowns, and are among the most useful and far-reaching devices of algebra.

ORAL EXERCISES

Evaluate the following determinants:

1. $\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}$.

2. $\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}$.

3. $\begin{vmatrix} 9 & -1 \\ 3 & -8 \end{vmatrix}$.

4. $\begin{vmatrix} 8 & 6 \\ -4 & 3 \end{vmatrix}$.

5. $\begin{vmatrix} -3 & -4 \\ 6 & -7 \end{vmatrix}$.

6. $\begin{vmatrix} a & -b \\ a & b \end{vmatrix}$.

7. $\begin{vmatrix} x & 7 \\ 7^2 & x^2 \end{vmatrix}$.

8. $\begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix}$.

9. $\begin{vmatrix} 3a & -2b \\ 4b & -5a \end{vmatrix}$.

10. $\begin{vmatrix} 6x & -2x^2 \\ -5x^3 & 6x^4 \end{vmatrix}$.

11. $\begin{vmatrix} 7a^2 & 2a \\ 4a & a^3 \end{vmatrix}$.

12. $\begin{vmatrix} 2xy & -y \\ x & 3 \end{vmatrix}$.

WRITTEN EXERCISES

Evaluate the following determinants:

1. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$.

2. $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$.

3. $\begin{vmatrix} 0 & 7 & 0 \\ 4 & 1 & 8 \\ 1 & 4 & 4 \end{vmatrix}$.

4. $\begin{vmatrix} 3 & 0 & 2 \\ 2 & 4 & 1 \\ 1 & 6 & 2 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$.

6. $\begin{vmatrix} -2 & 1 & 2 \\ -3 & 4 & -1 \\ 5 & 3 & 2 \end{vmatrix}$.

7. $\begin{vmatrix} 3 & -1 & 4 \\ 5 & -2 & 5 \\ 7 & -3 & 6 \end{vmatrix}$.

8. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 5 & 7 \end{vmatrix}$.

9. $\begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix}$.

10. $\begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ 3 & 1 & 0 \end{vmatrix}$.

11. $\begin{vmatrix} 4 & 2 & 0 \\ 7 & 9 & 6 \\ 0 & 4 & 2 \end{vmatrix}$.

12. $\begin{vmatrix} a & 3 & 2 \\ b & 5 & 6 \\ c & 7 & 0 \end{vmatrix}$.

13. $\begin{vmatrix} a-b & a & a+b \\ b-c & b & b+c \\ c-a & c & c+a \end{vmatrix}$.

Solve the following sets of equations by means of determinants:

14. $\begin{cases} 2x - y + z = 18, \\ x - 2y + 3z = 10, \\ 3x + y - 4z = 20. \end{cases}$

15. $\begin{cases} 5x - 3y + z = 15, \\ x - 3y - z = -3, \\ 2x - y + z = 8. \end{cases}$

16. $\begin{cases} 4x + 2y + z = 13, \\ x - y + z = 4, \\ x + 2y - z = 1. \end{cases}$

17. $\begin{cases} 6x + 4y - 4z = 4, \\ 4x - 2y + 8z = 0, \\ x + y + z = 4. \end{cases}$

18. $\begin{cases} x + 2y + 3z = 5, \\ 4x - 3y - z = 5, \\ x + y + z = 2. \end{cases}$

19. $\begin{cases} 2x - 8y + 3z = 2, \\ x - 4y + 5z = 1, \\ 3x - 10y - z = 5. \end{cases}$

20. $\begin{cases} x + y + z = 1, \\ x + 3y + 2z = 8, \\ 2x + 8y - 3z = 15. \end{cases}$

21. $\begin{cases} 2x - 3y + z = 5, \\ 3x + 2y - z = 5, \\ x + y + z = 3. \end{cases}$

22. $\begin{cases} x + y + z = 6, \\ 3x - 2y - z = 13, \\ 2x - y + 3z = 26. \end{cases}$

23. $\begin{cases} x + y + z = 6, \\ 4x - y - z = -1, \\ 2x + y - 3z = -6. \end{cases}$

24. $\begin{cases} 2x - 3y - 4z = -4, \\ 4x - 4y - 2z = 2, \\ x + y + z = 6. \end{cases}$

25. $\begin{cases} x + y + z = 0, \\ 5x + 3y + 4z = 2, \\ 2x - 2y + 3z = -8. \end{cases}$

26. $\begin{cases} x + 2y - z = 2, \\ 2x - y + z = 3, \\ x + 2y + z = 8. \end{cases}$

27. $\begin{cases} 2x - y - z = 6, \\ 2x - 2y + z = 10, \\ x + y - 3z = -2. \end{cases}$

28. $\begin{cases} x - y - z = 1, \\ 2x + 3y + z = 20, \\ x - 2y + z = 0. \end{cases}$

29. $\begin{cases} 5x - 2y + 3z = 5, \\ -2x + y - z = -1, \\ -x - y + 2z = 4. \end{cases}$

30. $\begin{cases} 4a - 3b + 2c = 1, \\ a + b - 4c = -7, \\ 7a - 4b + c = -2. \end{cases}$

31. $\begin{cases} 5m - 4n + r = -2, \\ 3m + n + 3r = 5, \\ 2m + 4n + 3r = 4. \end{cases}$

32. $\begin{cases} 9x - 4y - z = -4, \\ 2x + 5y - 6z = -12, \\ -x + 2y + 4z = 30. \end{cases}$

33. $\begin{cases} 4x + 7y + 3z = -4, \\ x - 3y + 2z = 4, \\ 5x + 2y + 4z = -4. \end{cases}$

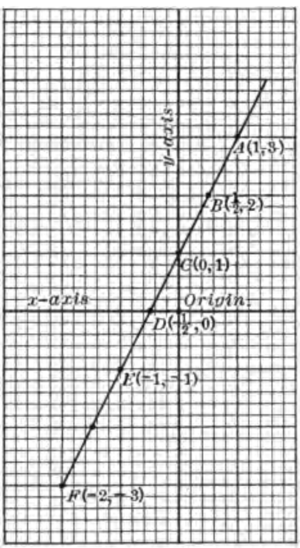
CHAPTER IX

GRAPHIC REPRESENTATION OF EQUATIONS

131. **Indeterminate Equations.** We have seen (§ 106) that a single equation in two unknowns has indefinitely many pairs of values of the unknowns which satisfy it.

Thus, in the equation $y = 2x + 1$, if we assign any value whatever to x, we find a corresponding value of y. *E.g.* if $x = 1$, $y = 2 \cdot 1 + 1 = 3$.

A picture or map of the solutions of an indeterminate equation in two unknowns may be made by means of the graph.



On a piece of square-ruled paper draw two lines at right angles to each other, called the **axes of coördinates.** Call their intersection the **origin.** Call the horizontal line the ***x*-axis,** and the vertical line the ***y*-axis.**

Now measure positive values of x to the *right* from the y-axis, and negative values to the *left;* and measure positive values of y *upward* from the x-axis, and negative values *downward.*

In this manner a point will be determined by each pair of values of x and y, as shown in the accompanying figure.

In the equation $y = 2x + 1$, we have for $x = 1$, $y = 2 \cdot 1 + 1 = 3$. Measuring one unit to the right of the y-axis, and three units up from the x-axis, we locate the point A.

Again, for $x = \frac{1}{2}$, we have $y = 2 \cdot \frac{1}{2} + 1 = 2$. Measuring $\frac{1}{2}$ unit to the right, and 2 units up, we locate the point B.

For $x = 0$, $y = 2 \cdot 0 + 1 = 1$. Measuring 1 unit up from the origin, we locate the point C.

For $x = -2$, $y = 2(-2) + 1 = -4 + 1 = -3$. Measuring 2 units to the left, and 3 units down, we locate the point F.

In this manner we may use as many pairs of values of x and y as we please, each satisfying the equation $y = 2x + 1$, and for each pair we locate a point.

132. The Graph of the First Degree Equation. If this work is done with care, it will be found that a straight line can be drawn passing through all the points thus located by the pairs of values of x and y which satisfy the equation $y = 2x + 1$.

This straight line is called the **graph** of the equation.

133. Coördinates. The values of x and y which locate a point are called the **coördinates of the point.** The x value is called the **abscissa** of the point, and the y value the **ordinate** of the point.

The coördinates of a point are usually written thus (x, y).

E.g. The coördinates of the point A in the graph are $(1, 3)$; those of B are $(\frac{1}{2}, 2)$; of C, $(0, 1)$; of D, $(-1, 0)$; of E, $(-1, -1)$; etc.

134. Linear Equations. An equation of the first degree in two variables is called a *linear equation*, since it can be shown that the graph of every such equation is a straight line.

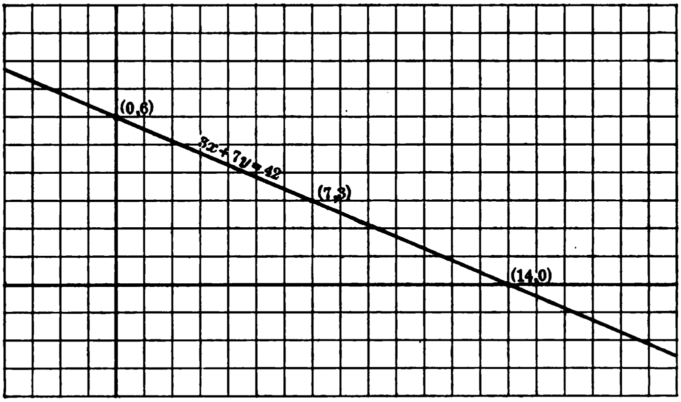
Thus, the equation $y = 2x + 1$, above, has the straight line AF extended indefinitely as its graph.

Since the graph of an equation of the first degree is a straight line, it is only necessary to locate *two* points in order to construct the graph.

E.g. In the equation $y = 2x + 1$, we need to find only the two points where the line cuts the axes. These are the easiest to find, since $y = 0$ where the line cuts the x-axis, and $x = 0$ where it cuts the y-axis.

Thus, we have $x = 0$, $y = 1$; and $y = 0$, $x = -\frac{1}{2}$.

135. Integral Solutions. It is often important to determine those solutions of an indeterminate equation which are *positive integers,* and for this purpose the graph is especially useful.



Example. Find the positive integral solutions of the equation

$$3x + 7y = 42.$$

Solution. Graph the equation carefully on cross-ruled paper, finding it to cut the x-axis at $x = 14$, $y = 0$, and the y-axis at $x = 0$, $y = 6$.

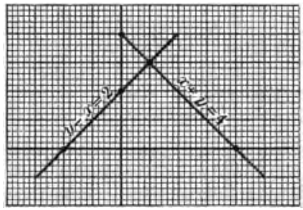
Look now for the *corner points* of the unit squares through which this straight line passes. The coördinates of these points, if there are any such, are the solutions required. In this case the line passes through only one such point whose coördinates are both positive, namely the point (7, 3). Hence the solution sought is $x = 7$, $y = 3$.

EXERCISES

Solve in positive integers by means of graphs, and check:

1. $x + y = 7$.
2. $x + y = 3$.
3. $x - 27 = -9y$.
4. $7y - 112 = -4x$.
5. $90 - 5x = 9y$.
6. $5x = 29 - 3y$.
7. $140 - 7x - 10y = 0$.
8. $8 - 2x - y = 0$.

136. Two Indeterminate Equations. In the case of two indeterminate equations, the coördinates of any point of intersection of their graphs form a solution of *both equations.*



If the equations are of the first degree the graphs are straight lines. Since these have *only one point* in common, there is only *one solution* of the given pair of equations.

E.g. On graphing $x + y = 4$ and $y - x = 2$, the lines are found to intersect in the point (1, 3). Hence the solution of this pair of equations is

$$x = 1, y = 3.$$

EXERCISES

Graph the following and thus find the solution of each pair of equations. Construct the graph of each equation by locating two points on it and drawing a straight line through them. Check by substituting in the equations.

1. $\begin{cases} 3x - 2y = -2, \\ x + 7y = 30. \end{cases}$

2. $\begin{cases} x + y = 2, \\ 3x + 2y = 3. \end{cases}$

3. $\begin{cases} x - 4y = 1, \\ 2x - y = -5. \end{cases}$

4. $\begin{cases} x = -1, \\ 2x - 3y = 1. \end{cases}$

5. $\begin{cases} 4x = 2y + 6, \\ x - 5 = y - 1. \end{cases}$

6. $\begin{cases} x = y - 5, \\ 5y = x + 9. \end{cases}$

7. $\begin{cases} 8x = 7y, \\ x + 3 = 5y + 3. \end{cases}$

8. $\begin{cases} y = 1, \\ 3y + 4x = y. \end{cases}$

9. $\begin{cases} 2x - 4y = 4, \\ x - y = 6y - 3. \end{cases}$

10. $\begin{cases} x = 4, \\ y + x = 8. \end{cases}$

11. $\begin{cases} y = -3, \\ 3x + 2y = 3. \end{cases}$

12. $\begin{cases} x = -2, \\ y = 5. \end{cases}$

INCONSISTENT AND DEPENDENT SYSTEMS

137. A pair of linear equations in two unknowns may be such that they either have *no solution* or have an *unlimited number of solutions.*

Example. Solve
$$\begin{cases} x - 2y = -2, & (1) \\ 3x - 6y = -12. & (2) \end{cases}$$

On graphing these equations they are found to represent two parallel lines. See Fig. 1. Since the lines have no point in common, it follows that the equations have no solution.

Attempting to solve them by means of determinants, § 124, we find:

$$x = \frac{\begin{vmatrix} -2 & -2 \\ -12 & -6 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix}} = \frac{-12}{0}, \quad y = \frac{\begin{vmatrix} 1 & -2 \\ 3 & -12 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix}} = \frac{-6}{0}.$$

But by § 31, $\frac{-12}{0}$ and $\frac{-6}{0}$ are *impossible operations*, and hence there are no values of x and y which satisfy these equations.

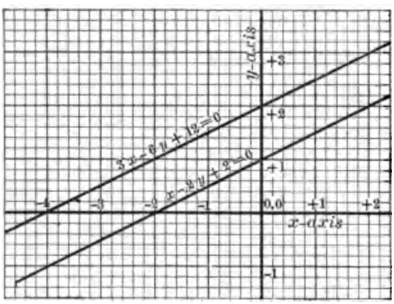


FIG. 1.

In this case, then, no solution is possible, and the equations are said to be **contradictory**.

Example 2. Solve
$$\begin{cases} 3x - 6y = -6, & (1) \\ x - 2y = -2. & (2) \end{cases}$$

On graphing these equations, they are found to represent the *same line*. Hence every pair of numbers satisfying one equation must satisfy the other also. See Fig. 2.

Solving these equations by determinants we find:

$$x = \frac{\begin{vmatrix} -6 & -6 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -6 \\ 1 & -2 \end{vmatrix}} = \frac{0}{0}, \quad y = \frac{\begin{vmatrix} 3 & -6 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -6 \\ 1 & -2 \end{vmatrix}} = \frac{0}{0}.$$

But by § 31, $\frac{0}{0}$ may represent *any number whatever*. In this case the solution is indeterminate and the equations are dependent; that is, one may be derived from the other.

Thus, (2) is derived from (1) by dividing both members by 3.

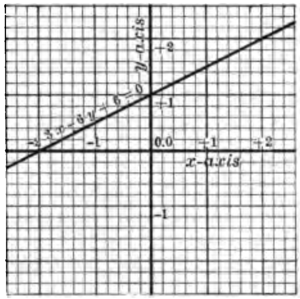


FIG. 2.

138. Test of Dependence by Determinants. The cases in which pairs of linear equations are *dependent* or *contradictory* are those in which the denominators of the expressions for x and y become *zero.* Hence, in order that such a pair of equations may have a *unique* solution, the determinant of the coefficients *must not reduce to zero.* This may be used as a test to determine whether a given pair of equations is independent and consistent.

Note. There is another sense in which the symbols $\frac{a}{0}$ and $\frac{0}{0}$ are used in higher mathematical works, but the above interpretation is the only one which may properly be given to them in elementary algebra.

WRITTEN EXERCISES

In the following, show both by determinants and by the graph, which pairs of equations are independent and consistent, which dependent, and which contradictory.

1. $\begin{cases} 5x - 3y = 5, \\ 5x - 3y = 9. \end{cases}$

2. $\begin{cases} x - 7 + 5y = y - x - 2, \\ 5x + 3y - 4 = 3x - y + 3. \end{cases}$

3. $\begin{cases} 7x - 3y - 4 = 2x - 2, \\ x + y - 3 = 2x - 7. \end{cases}$

4. $\begin{cases} x - 3y = 6, \\ 5x - 15y = 18. \end{cases}$

5. $\begin{cases} 3y - 4x - 1 = 2x - 5y + 8, \\ 2y - 5x + 8 = 3x + y. \end{cases}$

6. $\begin{cases} 3x - 6y + 5 = 2x - 5y + 7, \\ 5x + 3y - 1 = 3x + 5y + 3. \end{cases}$

7. $\begin{cases} 2y + 7x = 2 + 6x, \\ 4x - 3y = 4 + 3x - 5y. \end{cases}$

8. $\begin{cases} 5x - 3 = 7y + 8, \\ 2x + 7 = 4y - 9. \end{cases}$

9. $\begin{cases} 5x + 2y = 6 + 3x + 5y, \\ 3x + y = 18 - 3x + 10y \end{cases}$

10. $\begin{cases} 3x + 4y = 7 + 5y, \\ x - y = 6 - 2x. \end{cases}$

139. Contradictory and Dependent Systems of Three Equations. We have seen that in the case of two linear equations, the system is either *contradictory or dependent if the determinant of the coefficients is equal to zero.*

This statement is also true in the case of a system of three equations.

For example, in the system: $\begin{cases} x + 2y + 3z = 5, \\ 4x - 3y - z = 6, \\ 2x + 4y + 6z = 14. \end{cases}$

the determinant of the coefficients is

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & -3 & -1 \\ 2 & 4 & 6 \end{vmatrix} = -18 - 4 + 48 + 18 + 4 - 48 = 0.$$

Hence, no definite solution of this system exists, and no further attempt to solve the system need be made.

The solution by means of determinants may be extended to systems of four, five, six, etc., linear equations, and in all cases a definite solution can be found only when the *determinant of the coefficients is different from zero.*

HISTORICAL NOTE

Determinants. The theory of determinants forms one of the most remarkable phases of modern algebra. Like all other developments in algebraic methods, it grew slowly from small beginnings.

Gottfried Wilhelm Leibnitz (1646–1716) in Germany introduced the first notion of determinants in his effort to simplify the expressions arising in the solution of a set of linear equations; but little or no attention was given to the matter for another century.

Lagrange (1736–1813) in France used determinants of the third order in 1773 and proved that the square of a determinant is a determinant.

Vandermonde (1735–1796), also a Frenchman, was the first to give a connected and logical exposition of the theory of determinants.

Gauss (1777–1855) in Germany made systematic use of determinants, and it was he who first introduced the name *determinant*, though **Cauchy** (1789–1857) in France first brought the name into common use.

During the nineteenth century, a long list of mathematicians wrote on this fascinating theme. Possibly no topic in mathematics has attracted greater attention from so large a number of prominent writers.

Gottfried Wilhelm Leibnitz (1646–1716) was born in Leipsic and spent most of his life in Germany, though he lived several years in Paris and made an extended visit to England. Politics and diplomacy occupied much of his attention, but he found time to write extensively on philosophy, law, and mathematics.

Leibnitz was the first to introduce into mathematics the notion of a *determinant*.

CHAPTER X

RATIO VARIATION AND PROPORTION

RATIO AND VARIATION

140. Ratio. In many important applications fractions are called *ratios.*

E.g. $\frac{3}{5}$ is called the ratio of 3 to 5 and is sometimes written $3:5$.

A Ratio is a Number. It is to be understood that a ratio is the *quotient of two numbers* and hence is itself a *number.* We sometimes speak of the ratio of two magnitudes of the same kind, meaning thereby that these magnitudes are expressed in terms of a common unit and a ratio formed from the resulting *numbers.*

E.g. If the heights of two trees are 25 feet and 35 feet respectively, we say the *ratio of their heights* is $\frac{25}{35}$ or $\frac{5}{7}$.

141. Variables, Constants. In many problems, especially in physics, magnitudes are considered which are constantly changing. Number expressions representing such magnitudes are called *variables,* while those which represent fixed magnitudes are *constants.*

E.g. Consider a moving body. If t is the *number* of seconds from the time of starting and s the *number* of feet passed over in that number of seconds, then both the distance and the time are continually changing, and s and t are *variables.*

142. Constant Ratio. Two variables, such as s and t, may be such as to have a *fixed or constant ratio.*

E.g. If a body moves a distance of 5 feet in each second during its period of motion, then for any time t, the distance is $5t$. That is, $s = 5t$, or $s:t = 5$.

143. Direct Variation. When two variables are so related that for all pairs of corresponding values their ratio remains constant, then each one is said to *vary directly as the other.*

For example, if $s : t = k$ (a constant), then s varies directly as t. Also $t : s = \frac{1}{k}$ (likewise a constant). Hence t varies directly as s.

Variation is sometimes indicated by the symbol $\propto$. Thus $s \propto t$ means s varies as t, *i.e.* $\frac{s}{t} = k$, or $s = kt$.

As another illustration of direct variation, consider a rectangle with a fixed base b. Suppose the altitude is continually increased. Then the area A and the altitude h are *variables,* such that $A = b \cdot h$ or $A \div h = b$.

Hence, the area of a rectangle *varies directly as the altitude when the base is constant.*

Likewise, the area *varies directly as the base when the altitude is constant.*

144. Inverse Variation. When two variables are so related that for all pairs of corresponding values their product remains constant, then each one is said to vary *inversely as the other.*

E.g. Consider a rectangle whose area is A, and whose base and altitude are b and h respectively. Then, $A = b \cdot h$.

If, now, the base is multiplied by 2, 3, 4, etc., while the altitude is divided by 2, 3, 4, etc., then the area will remain constant. Hence, b and h may both vary while A remains constant.

The relation $bh = A$ may be written $b = A \times \frac{1}{h}$ or $h = A \times \frac{1}{b}$. Hence, either one is a constant times the reciprocal of the other. For this reason, one is said to vary *inversely as the other.*

If the temperature of a gas remains constant, then the pressure varies inversely as the volume. This is expressed by the equation $pv = k$ (where k is a constant.) This may also be written $p = \frac{k}{v}$ or $v = \frac{k}{p}$.

145. Other Forms of Variation. If $y = kx^2$, k being constant and x and y variables, then y varies *directly as* x^2.

The area of a circle is given by the equation $A = \pi r^2$. Hence, the area varies as the square of the radius.

If $y = kx^3$ then y varies directly as the cube of x, and x varies directly as the cube root of y.

The volume of a sphere is expressed by the equation $v = \frac{4}{3}\pi r^3$. Hence the volume of a sphere varies as the cube of the radius.

If $y = \frac{k}{x^2}$, then y *varies inversely as* x^2. If $y = k \cdot wx$, then *y varies jointly as w and x.*

If $y = \frac{k \cdot w}{x}$, then y varies *directly* as w and *inversely* as x.

The gravitational attraction A between two masses m_1 and m_2 at a distance d from each other is given by the equation $A = k\frac{m_1 m_2}{d^2}$ where k is a constant. Hence, their attraction varies jointly as their masses and inversely as the square of their distance apart.

Example. It is known that the resistance offered by a wire to an electric current passing through it varies directly as its length and inversely as the area of its cross section.

If a wire $\frac{1}{8}$ inch in diameter has a resistance of r units per mile, find the resistance of a wire $\frac{1}{4}$ inch in diameter and 3 miles long.

Solution. Let R represent the resistance of a wire of length l and cross-section area $s = \pi \cdot (\text{radius})^2$. Then $R = k \cdot \frac{l}{s}$ where k is some constant. Since $R = r$ when $l = 1$ mile and $s = \pi(\frac{1}{16})^2$, we have

$$r = k \cdot \frac{1}{\frac{\pi}{256}} \text{ or } k = \frac{\pi r}{256}.$$

Hence, when $l = 3$ and $s = \pi(\frac{1}{8})^2$, we have,

$$R = \frac{\pi r}{256} \cdot \frac{3}{\frac{\pi}{64}} = \frac{3}{4}r.$$

That is, the resistance of three miles of the second wire is $\frac{3}{4}$ the resistance *per mile* of the first wire.

PROBLEMS

1. If $z \propto w$, and if $z = 27$ when $w = 3$, find the value of z when $w = 4\frac{1}{8}$.

2. If z varies jointly as w and x, and if $z = 24$ when $w = 2$ and $x = 3$, find z when $w = 3\frac{1}{2}$ and $x = 7$.

3. If z varies inversely as w, and if $z = 11$ when $w = 3$, find z when $w = 66$.

4. If z varies directly as w and inversely as x, and if $z = 28$ when $w = 14$ and $x = 2$, find z when $w = 42$ and $x = 3$.

5. If z varies inversely as the square of w, and if $z = 3$ when $w = 2$, find z when $w = 6$.

6. If q varies directly as m and inversely as the square of d, and $q = 30$ when $m = 1$ and $d = \frac{6}{100}$, find q when $m = 3$ and $d = 600$.

7. If $y^2 \propto x^3$, and if $y = 16$ when $x = 4$, find y when $x = 9$.

8. The weight of a triangular plate of steel of uniform thickness varies jointly as its base and altitude. Find the base when the altitude is 4 and the weight 72, if it is known that the weight is 60 when the altitude is 5 and base 6.

9. The weight of a circular piece of steel cut from a sheet of uniform thickness varies as the square of its radius. Find the weight of a piece whose radius is 13 feet, if a piece of radius 7 feet weighs 196 pounds.

10. If a body starts falling from rest, its velocity varies directly as the number of seconds during which it has fallen. If the velocity at the end of 3 seconds is 96.6 feet per second, find its velocity at the end of 7 seconds; of ten seconds.

11. If a body starts falling from rest, the total distance fallen varies directly as the square of the time during which it has fallen. If in 2 seconds it falls 64.4 feet, how far will it fall in 5 seconds? In 9 seconds?

12. The number of vibrations per second of a pendulum varies inversely as the square root of the length. If a pendulum 39.1 inches long vibrates once in each second, how long is a pendulum which vibrates 3 times in each second?

13. Illuminating gas in cities is forced through the pipes by subjecting it to pressure in the storage tanks. It is found that the volume of a fixed amount of gas varies inversely as the pressure. A certain body of gas occupies 49,000 cu. ft. when under a pressure of 2 pounds per square inch. What space would it occupy under a pressure of $2\frac{4}{5}$ pounds per square inch?

14. The amount of heat received from a stove varies inversely as the square of the distance from it. A person sitting 15 feet from the stove moves up to 5 feet from it. By how much will this multiply the amount of heat received?

15. The weights of bodies of the same shape and of the same material vary as the cubes of corresponding dimensions. If a ball $3\frac{1}{4}$ inches in diameter weighs 14 oz., how much will a ball of the same material weigh whose diameter is $3\frac{1}{2}$ inches?

16. On the principle of problem 15, if a man 5 feet 9 inches tall weighs 165 pounds, what should be the weight of a man of similar build 6 feet tall?

17. The area of a circle varies as the square of its radius. If the area of a certain circle is 254.47 square inches and its radius is 9 inches, find the area of a circle whose radius is 11 inches. Also find the radius of a circle whose area is 212 square inches.

18. The time required for the vibration of a pendulum varies directly as the square root of the length. If a pendulum 39.1 inches long vibrates once in a second, find the time of vibration of a pendulum 54 inches long. Also find the length of a pendulum that vibrates once in 3 seconds.

PROPORTION

146. Definitions. The four numbers a, b, c, d, are said to be *proportional* or *to form a proportion*, if $\frac{a}{b}=\frac{c}{d}$. This is also sometimes written $a:b::c:d$, and is read *a is to b as c is to d.*

The four numbers are called the **terms of the proportion**; the first and fourth are the **extremes**, the second and third the **means** of the proportion. The first and third are the **antecedents** of the ratios, the second and fourth the **consequents.**

If a, b, c, x are proportional, x is the **fourth proportional** to a, b, c. If a, x, x, b are proportional, x is a **mean proportional** between a and b, and b is a **third proportional** to a and x.

147. Transforming a Proportion. A proportion may be transformed as shown in the following examples.

Example 1. If $a:b::c:d$ show that $a:c::b:d$.

Solution. From $\frac{a}{b}=\frac{c}{d}$, we are to show that $\frac{a}{c}=\frac{b}{d}$. Multiplying both members of $\frac{a}{b}=\frac{c}{d}$ by $\frac{b}{c}$, we have $\frac{a}{\cancel{b}}\cdot\frac{\cancel{b}}{c}=\frac{\cancel{c}}{d}\cdot\frac{b}{\cancel{c}}$ or $\frac{a}{c}=\frac{b}{d}$.

Example 2. If $a:b::c:d$, show that $a+b:b::c+d:d$.

Solution. Adding 1 to both members of $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a+b}{b}=\frac{c+d}{d}$.

3. From $a:b::c:d$ show how to derive $b:a::d:c$.
4. From $a:b::c:d$ show how to derive $a:c::b:d$.
5. From $a:b::c:d$ show how to derive $d:b::c:a$.
6. From $a:b::c:d$ derive $a-b:b::c-d:d$.
7. From $a:b::c:d$ derive $a+b:a-b::c+d:c-d$.

148. Names of Derived Proportions. The proportion $a:b::c:d$ is said to be taken by **alternation** when it is transformed into $a:c::b:d$; by **inversion** when it is transformed into $b:a::d:c$; by **composition** when it is transformed into $a+b:b::c+d:d$; by **division** when it is transformed into $a-b:b::c-d:d$; and by **composition and division** when it is transformed into $a+b:a-b::c+d:c-d$.

WRITTEN EXERCISES

1. What principles in the transformation of equations are involved (*a*) in taking a proportion by inversion, (*b*) by alternation.

2. From $a:b::c:d$ derive $a+b:a::c+d:c$.

3. From $a:b::c:d$ derive $a-b:a::c-d:c$.

4. From $a:b::c:d$ derive $a+c:b+d::a:b$.

5. From $a:b::c:d$ derive $a+b:c+d::b:d$.

6. From $a:b::c:d$ show that $a^2:b^2::c^2:d^2$.

7. From $a:b::c:d$ and $m:n::r:s$, show that $am:bn::cr:ds$.

8. From $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\cdots=\frac{a_n}{b_n}$ show that

$$\frac{a_1+a_2+\cdots+a_n}{b_1+b_2+\cdots+b_n}=\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\cdots=\frac{a_n}{b_n}.$$

Suggestion. Let $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\cdots=\frac{a_n}{b_n}=k$.

9. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{ma-b}{mc-d}=\frac{a}{c}$.

10. If $\frac{x}{y}=\frac{z}{w}$, show that $\frac{x+ky}{x-ky}=\frac{z+kw}{z-kw}$.

11. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{ma+nb}{ma-nb}=\frac{mc+nd}{mc-nd}$.

12. (*a*) Find a fourth proportional to 17, 19, and 187.
(*b*) Find a mean proportional between 6 and 54.
(*c*) Find a third proportional to 27 and 189.

13. If $ma=pq$, prove that the following proportions hold:
(1) $m:p::q:a$.
(2) $a:q::p:m$.

14. Find the unknown term in each of the following proportions:
(1) $x:42::27:126$; (2) $78:x::13:3$;
(3) $99:117::x:39$; (4) $171:27::57:x$.

ORAL EXERCISES

Solve the following equations:

1. $\frac{x}{4}=\frac{7}{2}$.
2. $\frac{x}{3}=\frac{5}{6}$.
3. $\frac{2x}{7}=\frac{5}{2}$.
4. $\frac{5}{x}=\frac{10}{7}$.
5. $\frac{4}{x}=\frac{7}{9}$.
6. $\frac{3}{7}=\frac{x}{16}$.
7. $\frac{4}{5}=\frac{2x}{15}$.
8. $\frac{3}{5}=\frac{6}{x}$.
9. $\frac{4}{7}=\frac{3}{2x}$.

WRITTEN EXERCISES

1. The area of a circle is πr^2, where r is the radius. Show that $\frac{a_1}{a_2}=\frac{r_1^2}{r_2^2}$, where a_1 and a_2 are the areas of any two circles whose radii are r_1 and r_2 respectively. Also show that $\frac{a_1}{a_2}=\frac{d_1^2}{d_2^2}$, where d_1 and d_2 are the diameters.

2. Compare the areas of two circular fields whose radii are 200 and 300 feet, respectively.

3. The volume of a sphere is $\frac{4}{3}\pi r^3$, where r is the radius. Show that $\frac{v_1}{v_2}=\frac{r_1^3}{r_2^3}$, where v_1 and v_2 are the volumes of any two spheres whose radii are r_1 and r_2, respectively. Also show that $\frac{v_1}{v_2}=\frac{d_1^3}{d_2^3}$, where d_1 and d_2 are the diameters.

4. Compare the volumes of two spheres whose radii are 2 and 5 inches, respectively.

5. The volumes of two spheres are in the ratio 8 to 27. Find the ratio of their radii.

6. The surface of a sphere is $4\pi r^2$, where r is the radius. Show that $\frac{s_1}{s_2}=\frac{r_1^2}{r_2^2}$, where s_1 and s_2 are the surfaces of two spheres whose radii are r_1 and r_2, respectively.

7. Compare the surfaces of two spheres whose radii are 2 and 5, respectively.

149. Variation Related to Proportion. A problem in variation may usually be stated in the form of a proportion:

For example, in the case of uniform motion, where the distance varies directly as the time, we have $s = kt$ or $\frac{s}{t} = k$. This simply means that if the body moves a distance s_1 in a time t_1 and a distance s_2 in a time t_2, that $\frac{s_1}{t_1} = \frac{s_2}{t_2}$, since both ratios are equal to the constant k. From this, by alternation we have $\frac{s_1}{s_2} = \frac{t_1}{t_2}$.

Again, it is known that if a body starts falling from rest, the distance fallen varies *directly as the square of the time* through which it has fallen. That is, $s = kt^2$, or $\frac{s}{t^2} = k$.

Hence, if the body falls a distance s_1 in a time t_1, and a distance s_2 in time t_2, we have the proportion $\frac{s_1}{t_1^2} = \frac{s_2}{t_2^2}$, since each ratio $= k$. From this proportion by alternation we have $\frac{s_1}{s_2} = \frac{t_1^2}{t_2^2}$.

This proportion may be obtained more directly from the equation $s = kt^2$ as follows: Since $s_1 = kt_1^2$ and $s_2 = kt_2^2$ then $\frac{s_1}{s_2} = \frac{kt_1^2}{kt_2^2} = \frac{t_1^2}{t_2^2}$.

Example. It is known that a body starting from rest falls a distance of 144.9 feet in 3 seconds. Find how far it will fall in 8 seconds.

Solution. Since the distance through which a body falls is proportional to the square of the time, we have

$$\frac{s_1}{s_2} = \frac{t_1^2}{t_2^2}.$$

In this example, we have $s_1 = 144.9$, $t_1 = 3$, and $t_2 = 8$. Required to find the distance s_2.

Hence,
$$\frac{144.9}{s_2} = \frac{3^2}{8^2},$$

or
$$s_2 = \frac{64 \times 144.9}{9} = 1030.4 \text{ feet.}$$

EXERCISES

1. The volume of a sphere varies directly as the cube of the radius. If a sphere whose radius is 4 inches contains 268.07 cubic inches, find the volume of a sphere whose radius is 6 inches.

2. When a weight is attached to a spring balance the index is displaced a distance proportional to the weight. Thus, if d_1 and d_2 are displacements and w_1, w_2 the corresponding weights, then $\frac{d_1}{d_2}=\frac{w_1}{w_2}$. If a 2-pound weight displaces the index $\frac{1}{4}$ inch, how much will a 50-pound weight displace it?

3. The intensity of light is *inversely* proportional to the square of the distance from the source of the light. That is, if i_1 and i_2 are the measures of intensities at the distances d_1 and d_2, then $\frac{i_1}{i_2}=\frac{d_2^2}{d_1^2}$. If the intensity of a given light at a distance of 2 feet is 20 candle power, find the intensity at 5 feet.

4. If w_1 and w_2 are weights resting on the two ends of a beam, and if the distances from the fulcrum are d_1 and d_2 respectively, then the beam will balance when $\frac{w_1}{w_2}=\frac{d_2}{d_1}$. That is, the weights are *inversely* proportional to the distances.

If a stone weighing 850 pounds at a distance of 1 foot from the fulcrum is to be balanced by a 50-pound weight, where should the weight be applied?

5. Find where the fulcrum should be in order that two boys weighing 110 and 80 pounds, respectively, may balance on the ends of a 16-foot plank.

6. The weight of a body above the earth's surface varies inversely as the square of its distance from the earth's center. If an object weighs 2000 pounds at the earth's surface, what would be its weight if it were 12,000 miles above the center of the earth, the radius of the earth being 4000 miles?

CHAPTER XI

POWERS AND ROOTS

150. Each of the operations thus far studied leads to a **single result.** See § 31.

E.g. Two numbers have one and only one *sum*, and one and only one *product*.

When a number is subtracted from a given number, there is one and only one *remainder*.

When a number is divided by a given number, there is one and only one *quotient*, provided the divisor is not zero.

We are now to study an operation which leads to **more than one result**; namely, the operation of finding roots.

Thus, both 3 and -3 are square roots of 9, since $3 \cdot 3 = 9$, and also $(-3)(-3) = 9$. The two square roots of 9 are written $\pm\sqrt{9} = \pm 3$.

Similarly a and $-a$ are both square roots of a^2, $x + y$ and $-x - y$ are both square roots of $x^2 + 2xy + y^2$.

151. Imaginary Numbers. There is no positive or negative number such that its square is negative.

Thus, $1^2 = +1$ and $(-1)^2 = +1$.

Hence, such an expression as $\sqrt{-1}$ has no meaning in terms of the numbers thus far used.

We now define the symbol $\sqrt{-1}$ by the equation

$$(\sqrt{-1})^2 = -1,$$

and we call $\sqrt{-1}$ the **imaginary unit.**

An *even* root of a negative number is called an **imaginary number.**

Thus, $\sqrt{-4}$, $\sqrt[4]{-2}$, are imaginary numbers.

All numbers which do not contain an imaginary number are called **real numbers.**

152. When irrational and imaginary numbers are admitted to the number system, it can be shown that every number has *two* square roots, *three* cube roots, *four* fourth roots, etc.

E.g. The square roots of 9 are $+3$ and -3. The square roots of -9 are $\pm\sqrt{-9} = \pm 3\sqrt{-1}$. The cube roots of 8 are 2, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$. The fourth roots of 16 are $+2$, -2, $+2\sqrt{-1}$ and $-2\sqrt{-1}$.

Any positive real number has two real roots of *even* degree, one positive and one negative.

E.g. ± 2 are fourth roots of 16. The square roots of 3 are $\pm\sqrt{3}$.

Any real number, positive or negative, has one real root of *odd* degree, whose sign is the same as that of the number itself.

E.g. $\sqrt[3]{27} = 3$, and $\sqrt[5]{-32} = -2$.

153. Principal Root. The positive *even* root of a positive real number, or the real *odd* root of any real number, is called the *principal root.*

The positive square root of a negative real number is also sometimes called the *principal* imaginary root.

E.g. 2 is the principal square root of 4; 3 is the principal 4th root of 81; -4 is the principal cube root of -64; and $+\sqrt{-3}$ is the principal square root of -3.

Unless otherwise stated the radical sign alone is understood to indicate the *principal root.*

Since a number has more than one root, it is necessary to limit certain theorems so as to make them apply to principal roots only.

Thus, the square root of 16 is not necessarily 4 unless the *principal* square root of 16 is understood.

Again, the cube root of 8 is not necessarily 2 unless the *principal* cube root is understood.

Unless this restriction is understood it may be easily shown that the conclusions in §§ 157, 159 are not true.

PRINCIPLES INVOLVING POWERS AND ROOTS

154. From § 44 $(2^3)^2 = (2^2)^3 = 2^6 = 64$.

In general, if n and k are any positive integers,

$$(b^k)^n = (b^n)^k = b^{nk}.$$

For $\qquad (b^k)^n = b^k \cdot b^k \cdot b^k \cdots \text{ to } n \text{ factors}$
$\qquad = b^{k+k+k\,\cdots\,\text{to } n \text{ terms}} = b^{nk}.$

Likewise, $\qquad (b^n)^k = b^n \cdot b^n \cdot b^n \cdots \text{ to } k \text{ factors}$
$\qquad = b^{n+n+n+\,\cdots\,\text{to } k \text{ terms}} = b^{kn}.$

Hence: *The nth power of the kth power of any base is the nkth power of that base.*

155. Again, we have $(2^3 \cdot 3^2)^2 = 2^6 \cdot 3^4$.

In general, if k, r, and n are any positive integers,

$$(a^k b^r)^n = a^{nk} b^{nr}.$$

For $\qquad (a^k b^r)^n = (a^k b^r) \cdot (a^k b^r) \cdots \text{ to } n \text{ factors.}$
$\qquad = (a^k \cdot a^k \cdots \text{ to } n \text{ factors})(b^r \cdot b^r \cdots \text{ to } n \text{ factors})$
$\qquad = (a^k)^n \cdot (b^r)^n = a^{nk} b^{nr}.$

Hence: *The nth power of the product of several factors is the product of the nth powers of those factors.*

ORAL EXERCISES

Remove the parenthesis in each of the following:

1. $(3^2)^3$.
2. $(2^3)^2$.
3. $(2 \cdot 5^2)^2$.
4. $(-2a^2)^3$.
5. $(3a^4)^2$.
6. $3(2x^3)^2$.
7. $(3 \cdot 2x^2)^2$.
8. $a^2(b^3)^2$.
9. $(-x^2y)^2$.
10. $(ab^2c)^2$.
11. $(a^3bc)^2$.
12. $(a^2b^3)^2$.
13. $(2ab^2)^3$.
14. $6(a^2b)^3$.
15. $a(bc^2)^4$.
16. $c(x^2y^3)^3$.
17. $c(a^2b)^2$.
18. $(-cx^2y^3)^3$.
19. $(ca^2b)^2$.
20. $(c^3a^2b)^3$.
21. $(-x^2y^3z^4)^2$.
22. $4(mn^3p^2)^3$.
23. $2(a^n)^k$.
24. $(2a^n)^k$.
25. $a^n(b^n)^k$.
26. $(a^nb^n)^k$.
27. $2(a^2b^4)^k$.
28. $(2a^2b^4)^k$.
29. $a^l(x^my^n)^k$.
30. $(a^lx^my^n)^k$.
31. $4(a^2b^3c^4)^{2n}$.
32. $2(a^{2a}y^{3b}z^{4b})^2$.
33. $(a^{3n}b^{2n}c^p)^q$.

156. From § 44, we have $\left(\frac{2^3}{3^2}\right)^2 = \frac{2^3}{3^2} \cdot \frac{2^3}{3^2} = \frac{2^6}{3^4}$.

In general, $$\left(\frac{a^k}{b^r}\right)^n = \frac{a^{nk}}{b^{nr}}.$$

For we have $$\left(\frac{a^k}{b^r}\right)^n = \frac{a^k}{b^r} \cdot \frac{a^k}{b^r} \cdot \frac{a^k}{b^r} \cdots \text{ to } n \text{ factors}$$
$$= \frac{a^{nk}}{b^{nr}}.$$

Hence: *The nth power of the quotient of two numbers equals the quotient of the nth powers of those numbers.*

ORAL EXERCISES

Remove the parenthesis in each of the following:

1. $\left(\frac{1}{2}\right)^2$.
2. $\left(\frac{1}{a}\right)^3$.
3. $\left(-\frac{1}{a^2}\right)^3$.
4. $\left(\frac{1}{a^3}\right)^2$.
5. $\left(\frac{a}{b^2}\right)^2$.
6. $\left(\frac{a^3}{b^4}\right)^2$.
7. $a^2\left(\frac{b}{a}\right)^2$.
8. $\left(\frac{a^2b}{a}\right)^2$.
9. $a^3\left(\frac{x^3}{a}\right)^2$.
10. $\left(-\frac{a^2x^3}{a}\right)^2$.
11. $4\left(\frac{xy}{2a}\right)^2$.
12. $\left(-\frac{xy}{a}\right)^2$.
13. $\left(\frac{x^ny^m}{a^l}\right)^k$.
14. $x^k\left(\frac{ab^2}{x}\right)^k$.
15. $\left(\frac{x^kab^2}{x}\right)^k$.

157. We may easily verify that $\sqrt{3^4} = 3^{4\div 2} = 3^2 = 9$.

In general, if k and r are positive integers and b any positive real number, we have:

$$\sqrt[r]{b^{kr}} = b^{kr\div r} = b^k.$$

For, from § 154, $(b^k)^r = b^{kr}$.

Hence, by definition b^k is an rth root of b^{kr}.

That is, $\sqrt[r]{b^{kr}} = b^k = b^{kr\div r}$.

Hence: *The rth root of the krth power of any positive real number is the kth power of that number.*

E g. $\sqrt[4]{2^{12}} = 2^{12\div 4} = 2^3 = 8$.

158. **Another general principle** is that, if a and b are any real numbers and r any positive integer, then

$$\sqrt[r]{ab} = \sqrt[r]{a} \cdot \sqrt[r]{b}.$$

For by § 155, $\qquad (\sqrt[r]{a} \cdot \sqrt[r]{b})^r = (\sqrt[r]{a})^r \cdot (\sqrt[r]{b})^r = a \cdot b,$

since the rth power of the rth root of a number is the number itself.

Hence, we have $\qquad ab = (\sqrt[r]{a} \cdot \sqrt[r]{b})^r.$

Taking the principal rth root of both members,

we have $\qquad \sqrt[r]{ab} = \sqrt[r]{a} \cdot \sqrt[r]{b}.$

Hence: *The rth root of the product of two positive real numbers equals the product of the rth roots of the numbers.*

ORAL EXERCISES

Remove the radical signs from each of the following:

1. $\sqrt{a^4}$.
2. $\sqrt{2^6}$.
3. $\sqrt[3]{-2^6}$.
4. $\sqrt[3]{a^6}$.
5. $\sqrt[4]{a^8}$.
6. $\sqrt{a^8}$.
7. $\sqrt[3]{-a^{12}}$.
8. $\sqrt[4]{a^{12}}$.
9. $\sqrt{a^4b^2}$.
10. $\sqrt[3]{a^3b^6c^9}$.
11. $\sqrt{a^{2n}}$.
12. $\sqrt{a^{2n}b^{4n}}$.
13. $\sqrt[3]{a^{6n}x^3}$.
14. $\sqrt[3]{-x^{3k}y^{6n}}$.
15. $\sqrt[4]{x^8y^{12n}}$.
16. $\sqrt[3]{-x^{12n}y^9}$.
17. $\sqrt[n]{a^{2n}b^{3n}}$.
18. $\sqrt[n]{a^{4n}b^nx^{kn}}$.
19. $\sqrt[2n]{x^{4n}y^{10n}}$.
20. $\sqrt[2n]{(a-b)^{4n}(a+b)^{2n}}$.
21. $\sqrt[3]{(x+y)^6(x-y)^9}$.

159. Again,

$$\sqrt[r]{\frac{a}{b}} = \frac{\sqrt[r]{a}}{\sqrt[r]{b}}.$$

For we have by § 156, $\left(\dfrac{\sqrt[r]{a}}{\sqrt[r]{b}}\right)^r = \dfrac{(\sqrt[r]{a})^r}{(\sqrt[r]{b})^r} = \dfrac{a}{b}.$

Hence, taking the principal rth root of the first and last members, we have

$$\frac{\sqrt[r]{a}}{\sqrt[r]{b}} = \sqrt[r]{\frac{a}{b}}.$$

That is: *The rth root of the quotient of two positive real numbers equals the quotient of the rth roots of the numbers.*

ORAL EXERCISES

Remove the radical sign from each of the following:

1. $\sqrt{\dfrac{1}{2^2}}$.
4. $\sqrt{\dfrac{a^2}{b^6c^2}}$.
7. $\sqrt[3]{-\dfrac{1}{a^6}}$.
10. $\sqrt{\dfrac{a^6}{b^9}}$.
2. $\sqrt{\dfrac{1}{a^2}}$.
5. $a\sqrt{\dfrac{a^2}{b^4}}$.
8. $\sqrt[3]{\dfrac{a^3}{b^6}}$.
11. $\sqrt[n]{\dfrac{a^{2n}}{b^{4n}}}$.
3. $\sqrt{\dfrac{a^4}{b^6}}$.
6. $\sqrt{\dfrac{a^4}{b^4}}$.
9. $a\sqrt[3]{-\dfrac{a^3}{b^9}}$.
12. $\sqrt[r]{\dfrac{(x+y)^r}{(x-y)^{2r}}}$.

160. It follows from §§ 154–159 that

(1) *Any positive integral power of a monomial is found by multiplying the exponents of the factors of the monomial by the exponent of the power.*

(2) *If a monomial is a perfect power of the* k*th degree, its* k*th root may be found by dividing the exponent of each factor by the index of the root.*

These principles may be expressed by the following formulas in which a and b are positive numbers, integral or fractional.

$$(a^k b^r)^n = a^{nk} b^{nr}.$$
$$\sqrt[r]{a^{nr} b^{mr}} = a^n b^m.$$

EXERCISES

Find the following indicated powers and roots, and reduce each expression to its simplest form:

1. $(a^3b^4c^5)^7$.
4. $(a^{x-y})^{x^2+xy+y^2}$.
7. $(3^5 \cdot 4^5 \cdot 2^5)^{a-b}$.
2. $(2^{a+b} \cdot 3^c \cdot 5^b)^{a-b}$.
5. $(x^4y^5z^{x+y})^{x-y}$.
8. $\sqrt[a]{3^{2a} \cdot 2^a \cdot 5^{3a}}$.
3. $\left(\dfrac{a^3b^4c^5}{2^3 \cdot 3^2 \cdot 4^3}\right)^2$.
6. $\left(\dfrac{5^2b^2mn}{3^7bc^4}\right)^3$.
9. $\sqrt[3]{\dfrac{-27 \cdot 8a^6}{64c^9d^{6a}}}$.
10. $(a^{m+n}b^{m-n}c^{mn})^{m+n}$.
13. $\sqrt[a-b]{3^{a^3-b^3} \cdot 4^{a-b} \cdot 5^{a^2-b^2}}$.
11. $(3^a \cdot 4^b \cdot 5^c)^{abx}$.
14. $\sqrt{64 \cdot 25 \cdot 256 \cdot 625}$.
12. $\sqrt[2a]{3^{6a} \cdot 4^{2a} \cdot 5^{8a}}$.
15. $\sqrt[3]{27 \cdot 125 \cdot 64 \cdot 3^{6a}}$.
16. $\sqrt{\dfrac{(a-b)^2(a^2+2ab+b^2)}{(a-b)^4(a+b)^2}}$.

SQUARE ROOT OF A POLYNOMIAL

161. The rule for finding the **square root of a polynomial** is derived from a study of the following squares.

From the squares

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a+b+c)^2 = [(a+b)+c]^2 = (a+b)^2 + 2(a+b)c + c^2,$$
$$(a+b+c+d)^2 = [(a+b+c)+d]^2$$
$$= (a+b+c)^2 + 2(a+b+c)d + d^2, \text{ etc.}$$

we see that

(1) If to a^2 we add $2ab + b^2 = (2a+b)b$, we get $(a+b)^2$;

(2) If to $(a+b)^2$ we add $2(a+b)c + c^2$
$= [2(a+b)+c]c$, we get $(a+b+c)^2$;

(3) If to $(a+b+c)^2$ we add $2(a+b+c)d + d^2$
$= [2(a+b+c)+d]d$, we get $(a+b+c+d)^2$, etc.

Hence, in squaring a polynomial:

*For every new **term** added to the root there is a new **part** added to the power. This new part consists of twice the sum of the **preceding** terms of the root plus the **last** term of the root, all multiplied by the last term of the root.*

This is expressed by the formula:

$$\boldsymbol{(a+b+c+d)^2}$$
$$\boldsymbol{= a^2 + (2a+b)b + [2(a+b)+c]c + [2(a+b+c)+d]d.}$$

WRITTEN EXERCISES

Write the following squares in the above form:

1. $(x+y+z)^2$.
2. $(x+y+z+v)^2$.
3. $(x+2y+3z+4v)^2$.
4. $(2a+b+3c+d)^2$.
5. $(b+4c+2d+e)^2$.
6. $(x+2a+3b+c)^2$.

Example 1. Find the square root of

$$9x^4 - 12x^3 + 28x^2 - 16x + 16.$$

Solution.

$$
\begin{array}{lr}
 & a \;+\; b \;+ c \\
\text{Square root (to be found).} & \underline{3x^2 - 2x + 4} \\
\text{Given square} = & 9x^4 - 12x^3 + 28x^2 - 16x + 16 \\
a^2 = (3x^2)^2 = & \underline{9x^4 \qquad\qquad\qquad\qquad\qquad\qquad} \\
2a = 2 \cdot 3x^2 = 6x^2 & -12x^3 + 28x^2 - 16x + 16 \\
(2a + b)b = (6x^2 - 2x)(-2x) = & \underline{-12x^3 + 4x^2 \qquad\qquad\quad} \\
2(a + b) = 6x^2 - 4x & 24x^2 - 16x + 16 \\
[2(a + b) + c]c = (6x^2 - 4x + 4) \cdot 4 = & \underline{24x^2 - 16x + 16} \\
 & 0
\end{array}
$$

Explanation. The first term of the root is $\sqrt{9x^4} = 3x^2$. The $9x^4$ is subtracted from the square.

The second term of the root is $-12x^3 \div (2 \cdot 3x^2) = -2x$.

The second part of the square is $(6x^2 - 2x)(-2x) = -12x^3 + 4x^2$, corresponding to $(2a + b)b$. This is now subtracted.

The third term of the root is $24x^2 \div (2 \cdot 3x^2) = 4$.

The third part of the square is $(6x^2 - 4x + 4) \cdot 4 = 24x^2 - 16x + 16$, corresponding to $[2(a + b) + c]c$. This is now subtracted.

Thus, at each step a new term of the root is found by dividing the first term of the remainder by twice the first term of the root; and then a new part of the power is built up and subtracted.

Since the final remainder is zero, the square root is exact.

Example 2. Find the square root of

$$16x^6 - 24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25.$$

Solution.

$$
\begin{array}{lr}
 & a \qquad + b \quad + c + d \qquad\qquad\qquad\qquad\qquad\qquad \\
\text{Square root} & \underline{4x^3 - 3x^2 + 2x - 5 \qquad\qquad\qquad\qquad\qquad\qquad} \\
\text{Given square} & 16x^6 - 24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25 \\
a^2 = (4x^3)^2 = & \underline{16x^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\
 & -24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25 \\
(2a + b)b = & \underline{-24x^5 + 9x^4 \qquad\qquad\qquad\qquad\qquad\qquad} \\
 & 16x^4 - 52x^3 + 34x^2 - 20x + 25 \\
[2(a + b) + c]c = & \underline{16x^4 - 12x^3 + 4x^2 \qquad\qquad\qquad} \\
 & -40x^3 + 30x^2 - 20x + 25 \\
[2(a + b + c) + d]\,d = & \underline{-40x^3 + 30x^2 - 20x + 25}
\end{array}
$$

162. From the preceding examples we have the following:

Rule. (1) *Arrange the polynomial according to ascending or descending powers of some letter.*

(2) *Find the square root of the first term, and write it as the first term of the root.*

(3) *Subtract the square of this first term of the root.*

(4) *Divide the first term of the remainder by twice the first term of the root, and write the quotient as the second term of the root.*

(5) *Add this second term of the root to twice the first term, and multiply the sum by the second term. This product is the second part of the square and is to be subtracted.*

(6) *Now use the sum of the first two terms of the root to find the third term, just as the first term was used to find the second; and continue in this manner till all the terms of the root are found.*

EXERCISES

Find the square roots of the following:

1. $m^2 + 4\,mn + 6\,ml + 4\,n^2 + 12\,ln + 9\,l^2.$
2. $4\,x^4 + 8\,ax^3 + 4\,a^2x^2 + 16\,b^2x^2 + 16\,ab^2x + 16\,b^4.$
3. $4\,x^6 - 12\,x^5 + 13\,x^4 - 14\,x^3 + 13\,x^2 - 4\,x + 4.$
4. $16\,a^6 + 24\,a^5 + 25\,a^4 + 20\,a^3 + 10\,a^2 + 4\,a + 1.$
5. $1 + 2\,x + 3\,x^2 + 4\,x^3 + 5\,x^4 + 4\,x^5 + 3\,x^6 + 2\,x^7 + x^8.$
6. $9\,a^2 - 6\,ab + 30\,ac + 6\,ad + b^2 - 10\,bc - 2\,bd + 25\,c^2 + 10\,cd + d^2.$
7. $9\,a^2 - 30\,ab - 3\,ab^2 + 25\,b^2 + 5\,b^3 + \frac{b^4}{4}.$
8. $\frac{4}{9}\,a^2x^4 - \frac{4}{3}\,abx^3z + \frac{8}{3}\,a^2bx^2z^2 + b^2x^2z^2 - 4\,ab^2xz^3 + 4\,a^2b^2z^4.$
9. $a^2 - 6\,ab + 10\,ac - 14\,ad + 9\,b^2 - 30\,bc + 42\,bd + 25\,c^2 - 70\,cd + 49\,d^2.$

10. $9a^6 - 24a^3b^4 - 18a^3c^5 + 6a^3d^2 + 16b^8 + 24b^4c^5 - 8b^4d^2 + 9c^{10} - 6c^5d^2 + d^4.$

11. $x^{10} - 8x^5w^5 + 16w^{10} - 4x^5y^3 + 16y^3w^5 + 4y^6 + 6x^5z^4 - 24z^4w^5 - 12y^3z^4 + 9z^8.$

12. $64l^{12} - 576l^{10} + 2160l^8 - 4320l^6 + 4860l^4 - 2916l^2 + 729.$

13. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$

14. $a^{18} + 12a^{15} + 60a^{12} + 160a^9 + 240a^6 + 192a^3 + 64.$

THE SQUARE ROOT OF AN ARITHMETIC NUMBER

163. Rule for Finding the First Term of the Square Root of an Integral Number. Since $1^2 = 1$ and $9^2 = 81$, the square of a number of one figure contains either *one* or *two* figures.

Since $10^2 = 100$ and $99^2 = 9801$, the square of a number of two figures contains either *three* or *four* figures.

Similarly, the square of a number of three figures contains either *five* or *six* figures, and so on.

Rule. Hence, to find the first figure in the root:

(1) *Separate the number into groups of two figures each, counting from units' place toward the left. The last group may contain only one figure.*

(2) *Take the square root of the largest square in the left-hand group; this is the first figure of the root. There are as many figures in the root as there are groups in the number.*

Examples. 1. To find the first figure and the number of figures in the square root of 450,769 we write it thus, 45 07 69.

Since there are three groups of two figures each, the square root contains three figures, and hence it starts with a digit in hundreds' place.

Since 36 is the largest square in 45, the first figure in the root is $\sqrt{36} = 6$.

2. Similarly, the square root of 6,762,436, written 6 76 24 36, contains four figures, of which the first one is in thousands' place.

Since 4 is the largest square in 6, the first figure of the root is $\sqrt{4} = 2$.

ORAL EXERCISES

Give the first figure in the square root of each of the following, and state whether it stands in units', tens', or hundreds' place:

1. 8947.	5. 90,401.	9. 7347.	13. 107.
2. 6205.	6. 63,401.	10. 73,470.	14. 4091.
3. 19,140.	7. 1428.	11. 14,051.	15. 10,007.
4. 72,048.	8. 194,670.	12. 140,051.	16. 100,007.

164. The square root of an arithmetic number may be found by the process used for polynomials in § 162.

Illustrative Example 1. Find the square root of 405769.

Solution.

	SQUARE	SQUARE ROOT	
		$a + b + c$	
	40 57 69	$600 + 30 + 7 = 637$	
$a^2 = 600^2 =$	36 00 00		
$2a = 1200$	4 57 69		
$b = 30$			
$2a + b = 1230$	3 69 00		$= (2a + b)b$
$2(a + b) = 1260$	88 69		
$c = 7$			
$2(a + b) + c = 1267$	88 69		$= [2(a + b) + c]c$
	0		

Explanation. The first figure in the root is the square root of the largest square in the left-hand group, and since there are three groups, the root starts with 600, which corresponds to a of the formula (§ 161).

Subtracting the square of 600, we have a remainder 4 57 69.

The trial divisor is $2a = 1200$ and when 4 57 69 is divided by 1200, the largest number of tens in the quotient is 3. Hence 30 corresponds to b of the formula.

The complete divisor is $2a + b = 1230$.

The next trial divisor is $2(a + b) = 1260$ and $8869 \div 1260$ gives 7 as the largest number of units. This is c of the formula.

165. In case a square consists of a **whole number and a decimal part**, the figures in the *integral part* of the square root are found exactly as in Example 1, page 121. To find the *decimal part* of the root, we proceed as in the next illustrative example.

ORAL EXERCISES

Give the first figure in the square root of each of the following, and tell in which place it stands:

1. 12.645.	4. 941.61.	7. 49.29.
2. 1.2645.	5. 94.16.	8. 4.929.
3. 126.45.	6. 9.416.	9. 492.9.

Illustrative Example 2. Find the square root of 67.7329.

Solution.

	SQUARE	SQUARE ROOT
		$a + b + c$
	67.73 29	$8 + .2 + .03 = 8.23$
$a^2 = \quad 8^2 =$	64	
$2\,a = 16$	3.73 29	
$b = \quad .2$		
$2\,a + b = 16.2$	3.24	$=(2\,a + b)b$
$2(a + b) = 16.4$	.49 29	
$c = \quad .03$		
$2(a + b) + c = 16.48$	.49 29	$=[2(a + b) + c]c$
	0	

Explanation. There is only one group of two figures to the left of the decimal point. Hence the first figure of the root is units' place.

In getting the second figure of the root, the trial divisor is $2\,a = 16$. The quotient is $b = .2$ since $.2 \times 16 = 3.2$. The quotient could not be .3 since $.3 \times 16 = 4.8$.

Similarly, in getting the third figure, we divide .4929 by 16.4 and the quotient is .03 since $.03 \times 16.4 = .492$.

Since the square of a decimal contains twice as many decimal places as the number itself, there will be one decimal figure in the root for every two in the square.

Illustrative Example 3. Find the square root of 9.1204.

Solution.

	SQUARE	SQUARE ROOT
		$a + b$
	9.12 04	$3 + .02 = 3.02$
$a^2 = 3^2 =$	9	
$2a = 6$	.12 04	
$b = .02$		
$2a + b = 6.02$	.12 04	$= (2a + b)b$
	0	

Explanation. Since there is only one group to the left of the decimal point, the first figure of the root is in units' place.

In this case, in dividing .1204 by 6, the quotient is .02 since $.02 \times 6 = .12$; that is, there is a zero in tenths' place, and there are only two terms in the root.

166. The Trial Divisor. From these examples we see that

(1) *Any trial divisor is twice that part of the root which has already been found.*

(2) *Any term of the root is found by dividing the last remainder by the trial divisor.*

(3) *In taking a quotient as a term of the root, allowance must be made because the last term is to be added to the trial divisor to form the complete divisor.*

EXERCISES

Find the square root of each of the following:

1. 294,849.	7. 1849.	13. 357.21.
2. 37,636.	8. 73,441.	14. 16,641.
3. 872,356.	9. 100,489.	15. 32,761.
4. 599,076.	10. 265.69.	16. 2332.89.
5. 3481.	11. 87.4225.	17. 1197.16.
6. 7569.	12. 170,569.	18. 6272.64.

167. The First Digit in the Square Root of a Decimal Number. In case a number has no integral part, the first term of its square root is found as in the following examples:

1. In .17 42 the first digit in the root is .4 since the square of .4 is .16, the largest square in .17.
2. In .05 42 the first digit in the root is .2 since the square of .2 is .04, the largest square in .05.
3. In .00 70 the first digit in the root is .08 since the square of .08 is .0064, the largest square in .0070.
4. In .00 07 the first digit in the root is .02 since the square of .02 is .00 04, the largest square in .0007.
5. In .00 00 69 the first digit in the root is .008 since the square of .008 is .000064, the largest square in .000069.

168. From these examples we have the following

Rule. To find the first digit in the square root of a decimal number:

(1) *Divide the number into groups of two figures each, counting from the decimal point toward the right, adding a zero if necessary to complete the last group.*

(2) *Take the square root of the largest square contained in the first group which is not all zeros, and prefix to it as many zeros as there are complete groups of zeros.*

Illustrative Example. Find the square root of .06783.

Solution.		SQUARE	SQUARE ROOT
			$a+\quad b+\quad c$
		.06 78 30	.2 + .06 + .0004
$a^2 = .2^2$		$= .04$	
$2a = 2 \times .2 =$	.4	.02 78	
$b =$	.06		
$2a + b =$	.46	.02 76	$=(2a+b)b$
$2(a+b) =$	.52	.00 02 30 00	
$c =$	.0004		
$2(a+b)+c =$	.5204	.00 02 08 16	$=[2(a=b)+c]c$
		.00 00 21 84	

Explanation. According to the rule, .2 is the first term of the root because .04 is the largest square in .06 and there is no group preceding .06. The process is the same as in the case of an integral square, but special care is now needed in handling the decimal points, which is done exactly as in operations upon decimals in the process of division in arithmetic.

For instance, in finding the third term in this example, we divide .00023 by $2(.26) = .52$ and the quotient lies between .0004 and .0005. Hence $c = .0004$. Zeros are annexed to .00023 to correspond to the number of decimal places in the product $.5204 \times .0004$.

The three terms of the root thus found are $.2 + .06 + .0004 = .2604$.

To find the next term of the root we would divide .00002184 by $2(.2604) = .5208$, finding the quotient .00004. We would then add .00004 to .5208 and multiply the sum by .00004, annexing zeros to the dividend as before.

169. Approximate Square Roots. Evidently the process in this example may be carried on indefinitely. .2604 is an **approximation** to the square root of .06783; in fact, the square of .2604 differs from .06783 by only .00002184. The nearest approximation using three decimal places is .260. If the fourth figure were 5, or any digit greater than 5, then .261 would be the nearest approximation using three decimal places. Hence, four places must be found in order to be sure of the nearest approximation to three places; and five places must be found in order to be sure of the nearest approximation to four places, and so on.

Find the square root of each of the following, correct to two significant figures:

1. 387.	**9.** 5.	**17.** .2.	**25.** .0067.
2. 5267.	**10.** 7.	**18.** .3.	**26.** .0091.
3. 2.92.	**11.** 8.	**19.** .45.	**27.** .00087.
4. 27.29.	**12.** 11.	**20.** .05.	**28.** .000191.
5. 51.	**13.** .02.	**21.** .6.	**29.** .000471.
6. 3.824.	**14.** .003.	**22.** .7.	**30.** .00186.
7. 2.	**15.** .5.	**23.** .8.	**31.** .7004.
8. 3.	**16.** .005.	**24.** .9.	**32.** .0981.

170. Cube and Square Roots Compared. We have seen that the process for finding the **square root** of a polynomial is obtained by studying the relation of the expressions $a+b$, $a+b+c$, $a+b+c+d$, etc., to their respective squares.

The process for finding the **cube root** of a polynomial is obtained by studying the relation of the cubes,

$$a^3+3a^2b+3ab^2+b^3 \text{ or } a^3+(3a^2+3ab+b^2)b,$$

$$a^3+(3a^2+3ab+b^2)b+[3(a+b)^2+3(a+b)c+c^2]c,$$

and $$a^3+(3a^2+3ab+b^2)b+[3(a+b)^2+3(a+b)c+c^2]c$$
$$+[3(a+b+c)^2+3(a+b+c)d+d^2]d,$$

to their cube roots, $a+b$, $a+b+c$, and $a+b+c+d$.

An example will illustrate the process.

Example 1. Find the cube root of

$$27m^3+108m^2n+144mn^2+64n^3.$$

Given cube,	$27m^3+108m^2n+144mn^2+64n^3$	$a+b$
$a^3=(3m)^3=$	$27m^3$	$3m+4n$, cube root
$3a^2=3(3m)^2=27m^2$	$108m^2n+144mn^2+64n^3=$	1st remainder
$3ab=3(3m)(4n)=36mn$		
$b^2=(4n)^2=16n^2$		
$3a^2+3ab+b^2=27m^2+36mn+16n^2$	$108m^2n+144mn^2+64n^3$	$=(3a^2+3ab+b^2)b$
	0	

Explanation. The cube root of the first term, namely $3m$, is the first term of the root and corresponds to a of the formula. Cubing $3m$ gives $27m^3$, which is the a^3 of the formula.

Subtracting $27m^3$ leaves $108m^2n+144mn^2+64n^3$, which is the $(3a^2+3ab+b^2)b$ of the formula.

Since b is not yet known, we cannot find completely either factor of $(3a^2+3ab+b^2)b$, but since a has been found, we can get the first term of the factor $3a^2+3ab+b^2$; viz., $3a^2$ or $3(3m)^2=27m^2$, which is the partial divisor. Dividing $108m^2n$ by $27m^2$ we have $4n$, which is the b of the formula.

Then $3a^2+3ab+b^2=3(3m)^2+3(3m)(4n)+(4n)^2=27m^2+36mn+16n^2$ is the complete divisor. This expression is then multiplied by $b=4n$, giving $108m^2n+144mn^2+64n^3$, which corresponds to $(3a^2+3ab+b^2)b$ of the formula. On subtracting, the remainder is zero and the process ends. Hence, $3m+4n$ is the required root.

Example 2. Find the cube root of

$$33x^4 - 9x^5 + x^6 - 63x^3 + 66x^2 - 36x + 8.$$

We first arrange the terms with respect to the exponents of x.

	$a + \quad b + c$
	$x^2 - 3x + 2$, cube root
Given cube,	$x^6 - 9x^5 + 33x^4 - 63x^3 + 66x^2 - 36x + 8$
$a^3 = (x^2)^3 =$	x^6
$3a^2 = 3(x^2)^2 = 3x^4$	$-9x^5 + 33x^4 - 63x^3 + 66x^2 - 36x + 8$
$3a^2 + 3ab + b^2 = 3x^4 - 9x^3 + 9x^2$	$-9x^5 + 27x^4 - 27x^3$
$3(a+b)^2 = 3(x^2 - 3x)^2 = 3x^4 - 18x^3 + 27x^2$	$6x^4 - 36x^3 + 66x^2 - 36x + 8$
$3(a+b)^2 + 3(a+b)c + c^2 = 3x^4 - 18x^3 + 33x^2 - 18x + 4$	$6x^4 - 36x^3 + 66x^2 - 36x + 8$
	0

The cube root of x^6, or x^2, is the first term of the root. The first partial divisor, which corresponds to $3a^2$ of the formula, is $3(x^2)^2 = 3x^4$. Dividing $-9x^5$ by $3x^4$ we have $-3x$, which is the second term of the root, corresponding to b of the formula.

After these two terms of the root have been found, we use $3(a+b)^2 = 3(x^2 - 3x)^2 = 3x^4 - 18x^3 + 27x^2$ as the new partial divisor and find that the next term of the root is 2.

The complete divisor is $3(a+b)^2 + 3(a+b)c + c^2 = 3x^4 - 18x^3 + 33x^2 - 18x + 4$. On multiplying this expression by 2 and subtracting, the remainder is zero. Hence the root is $x^2 - 3x + 2$.

In case there are four terms in the root, the next partial divisor is $3(a+b+c)^2$ and $3(a+b+c)^2 + 3(a+b+c)d + d^2$ is the complete divisor. The process is then precisely the same as in the preceding step.

In practice, the next term of the root is always found by dividing the remainder by three times the square of the first term of the root.

ORAL EXERCISES

1. How are the terms of a polynomial arranged for the extraction of its cube root?

2. How is the first term of the root found?

3. How is the first partial divisor found? the second?

4. How is the second term of the root found? the third?

WRITTEN EXERCISES

Find the cube roots of each of the following:

1. $8x^3 - 36x^2y + 54xy^2 - 27y^3$.
2. $8a^3 - 12a^2b + 6ab^2 - b^3$.
3. $1728x^6 + 1728x^4y^3 + 576x^2y^6 + 64y^9$.
4. $a^3 + 3a^2b + 3a^2 + 3b^2 + 3ab^2 + 6ab + 3a + 3b + b^3 + 1$.
5. $x^3 + y^3 + z^3 + 3xy^2 + 3xz^2 + 3x^2y + 3x^2z + 3y^2z + 3yz^2 + 6xyz$.
6. $x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3$.
7. $a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3$.
8. $8x^6 - 36x^5 + 114x^4 - 207x^3 + 285x^2 - 225x + 125$.
9. $27z^6 - 54az^5 + 63a^2z^4 - 44a^3z^3 + 21a^4z^2 - 6a^5z + a^6$.
10. $1 - 9y^2 + 39y^4 - 99y^6 + 156y^8 - 144y^{10} + 64y^{12}$.
11. $125x^6 - 525x^5y + 60x^4y^2 + 1547x^3y^3 - 108x^2y^4 - 1701xy^5 - 729y^6$.
12. $64l^{12} - 576l^{10} + 2160l^8 - 4320l^6 + 4860l^4 - 2916l^2 + 729$.
13. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.
14. $a^9 - 9a^8b + 36a^7b^2 - 84a^6b^3 + 126a^5b^4 - 126a^4b^5 + 84a^3b^6 - 36a^2b^7 + 9ab^8 - b^9$.
15. $a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc + 3ac^2 + 8b^3 - 12b^2c + 6bc^2 - c^3$.
16. $343a^6 - 441a^5b + 777a^4b^2 - 531a^3b^3 + 444a^2b^4 - 144ab^5 + 64b^6$.
17. $a^{18} + 12a^{15} + 60a^{12} + 160a^9 + 240a^6 + 192a^3 + 64$.
18. $27l^{12} + 189l^{11} + 198l^{10} - 791l^9 - 594l^8 + 1701l^7 - 729l^6$.

ROOTS OF NUMBERS EXPRESSED IN ARABIC FIGURES

171. The cube root of a number expressed in Arabic figures may be found by the process used for polynomials as in the case of square root, pp. 120–125. An example will illustrate the process.

Example 1. Find the cube root of 389,017.

In order to decide how many digits there are in the root, we observe that $10^3 = 1000$, $100^3 = 1,000,000$. Hence the cube root of 389,017 lies between 10 and 100, that is, it contains two digits. Since $70^3 = 343,000$ and $80^3 = 512,000$, it follows that 7 is the largest number possible in tens' place. The work is arranged as follows:

		$a + b$
The given cube,	389 017	$70 + 3$, cube root.
$a^3 = 70^3 =$	343 000	
$3a^2 = 3 \cdot 70^2 = 14700$	46 017	1st remainder.
$3ab = 3 \cdot 70 \cdot 3 = 630$		
$b^2 = 3^2 = 9$		
$3a^2 + 3ab + b^2 = 15339$	46 017	$= (3a^3 + 3ab + b^2)b$.
	0	

Having decided as above that the a of the formula is 7 tens, we cube this and subtract, obtaining 46,017 as the remaining part of the power.

The first partial divisor, $3a^2 = 14,700$, is divided into 46,017, giving a quotient 3, which is the b of the formula. Hence, the first complete divisor, $3a^2 + 3ab + 3b^2$, is 15,339 and the product, $(3a^2 + 3ab + b^2)b$, is 46,017. Since the remainder is zero, the process ends and 73 is the cube root sought.

172. The cube of any number from 1 to 9 contains one, two, or three digits; the cube of any number between 10 and 99 contains four, five, or six digits; the cube of any number between 100 and 999 contains seven, eight, or nine digits, etc. Hence it is evident that if the digits of a number are separated into groups of three figures each, counting from units' place toward the left, the number of groups thus formed is the same as the number of digits in the root.

The left-hand group may contain one, two, or three digits, as the case may be.

Example 2. Find the cube root of 13,997,521.

$$
\begin{array}{rr|l}
 & & \quad a + b \;\; + c \\
\text{The given cube,} & 13\;997\;521 & \underline{200 + 40 + 1 = 241,} \text{ cube root.} \\
a^3 = 200^3 = & \underline{8\;000\;000} & \\
3a^2 = 120000 & 5\;997\;521 & \\
3ab = \;\;24000 & & \\
b^2 = \underline{\;\;\;1600} & & \\
145600 & \underline{5\;824\;000} = (3a^2 + 3ab + b^2)b & \\
3(a+b)^2 = 172800 & 173\;521 & \\
3(a+b)c = \;\;\;720 & & \\
c^2 = \underline{\;\;\;\;\;1} & & \\
173521 & \underline{173\;521} = [3(a+b)^2 + 3(a+b)c + c^2]c. & \\
 & 0 &
\end{array}
$$

Since the root contains three digits, the first one is the cube root of 8, the largest integral cube in 13.

The first partial divisor, $3 \cdot 200^2 = 120{,}000$, is completed by adding $3\,ab = 3 \cdot 200 \cdot 40 = 24{,}000$, and $b^2 = 1600$.

The second partial divisor, $3(a + b)^2$, which stands for $3(200 + 40)^2 = 172{,}800$, is completed by adding $3(a + b)c$ which stands for $3 \cdot 240 \cdot 1 = 720$, and c^2 which stands for 1^2. At this step the remainder is zero and the root sought is 241.

EXERCISES

Find the square root of each of the following:

1. 58,081.	2. 795,664.	3. 11,641,744.

Find the cube root of each of the following:

4. 110,592.	7. 205,379.	10. 2,146,689.
5. 571,787.	8. 31,855,013.	11. 19,902,511.
6. 7,301,384.	9. 5,929,741.	12. 817,400,375.

173. Since the cube of a decimal fraction has three times as many places as the given decimal, it is evident that the cube root of a decimal fraction contains one decimal place for every three digits in the cube. Hence for the purpose of determining the places in the root, the decimal part of a cube should be separated into groups of three digits each, counting from the decimal point toward the right.

Example. Approximate the cube root of 34.567 to two places of decimals.

		$a+b+c+d$
	34.567	$3+.2+.05+.007=3.257$
$a^3=3^3=$	27.000	
$3a^2=3\cdot 3^2=27$	7.567	
$3ab=3\cdot 3(.2)=1.8$		
$b^2=(.2)^2=.04$		
28.84	5.768	$=(3a^2+3ab+b^2)b$
$3a'^2=3(3.2)^2=30.72$	1.799000	
$3a'c=3(3.2)(.05)=.48$		
$c^2=(.05)^2=.0025$		
31.2025	1.560125	$=(3a'^2+3a'c'+c^2)c$
$3a''^2=3(3.25)^2=31.6875$	.238875000	
$3a''d=3(3.25)(.007)=.06825$		
$d^2=(.007)^2=.000049$		
31.755799	.222290593	$=(3a''^2+3a''d+d^2)d$
	.016584407	

For the sake of brevity a' is used for $a+b$ and a'' for $a+b+c$. The decimal points are handled exactly as in arithmetic work.

174. Evidently the above process can be carried on indefinitely. 3.257 is an **approximation** to the cube root of 34.567. In fact the cube of 3.257 differs from 34.567 by less than the small fraction .017. The nearest approximation using two decimal places is 3.26. If the third decimal place were any digit less than 5, then 3.25 would be the nearest approximation using two decimal places. Hence, three places must be found in order to be sure of the nearest approximation to two places.

EXERCISES

Approximate the cube root of each of the following to two places of decimals:

1. 21.4736.	**4.** 2.	**7.** .3917.	**10.** 6410.37.	**13.** 572.274.
2. 6.5428.	**5.** 3.	**8.** .5.	**11.** .004178.	**14.** 31.7246.
3. 58.	**6.** .003.	**9.** .05.	**12.** 200.002.	**15.** 54913.416.

CHAPTER XII

EXPONENTS AND RADICALS

FRACTIONAL AND NEGATIVE EXPONENTS

175. The meaning heretofore attached to the word *exponent* cannot apply to a fractional or negative number.

E.g. Such an exponent as $\frac{3}{4}$ or -5 cannot indicate the *number of times* a base is used as a factor.

It is possible, however, to interpret fractional and negative exponents in such a way *that the laws of operations which govern positive integral exponents shall apply to these also.*

176. The laws for positive integral exponents are:

I.	$a^m \cdot a^n = a^{m+n}$.	§ 44
II.	$a^m \div a^n = a^{m-n}$.	§ 46
III.	$(a^m)^n = a^{mn}$.	§ 154
IV.	$(a^m \cdot b^n)^p = a^{mp}b^{np}$.	§ 155
V.	$(a^m \div s^n)^p = a^{mp} \div s^{np}$.	§ 156

177. Fractional Exponents. Assuming Law I to hold for positive fractional exponents and letting r and s be positive integers, we determine as follows the meaning of $b^{\frac{r}{s}}$ (read b *exponent* r *divided by* s).

By definition, $\left(b^{\frac{r}{s}}\right)^s = b^{\frac{r}{s}} \cdot b^{\frac{r}{s}} \cdots$ to s factors,

which by Law I $= b^{\frac{r}{s}+\frac{r}{s}+\cdots \text{ to } s \text{ terms}} = b^{s\times\frac{r}{s}} = b^r$.

Hence, $b^{\frac{r}{s}}$ is one of the s equal factors of b^r.

That is, $b^{\frac{r}{s}} = \sqrt[s]{b^r}$, and in particular $b^{\frac{1}{s}} = \sqrt[s]{b}$.

Hence, $b^{\frac{1}{n}}$ is an nth root of b.

Similarly, from $\left(b^{\frac{1}{s}}\right)^r = b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdots$ to r factors, $= b^{\frac{r}{s}}$,

we show that $\quad b^{\frac{r}{s}} = \left(b^{\frac{1}{s}}\right)^r = (\sqrt[s]{b})^r.$

Hence, $\quad b^{\frac{r}{s}} = \sqrt[s]{b^r} = (\sqrt[s]{b})^r.$

Thus, a positive fractional exponent means *a root of a power or a power of a root, the numerator indicating the power and the denominator indicating the root.*

E.g. $a^{\frac{2}{3}} = \sqrt[3]{a^2} = (\sqrt[3]{a})^2$; $8^{\frac{2}{3}} = \sqrt[3]{64} = 4$, or $(\sqrt[3]{8})^2 = 2^2 = 4$.

178. Zero Exponents. Assuming that Law I holds for zero exponents, and letting t be any positive number, we have

$$b^t \cdot b^0 = b^{t+0} = b^t.$$

Hence, $\quad b^0 = \dfrac{b^t}{b^t} = 1.$

It follows that any number with the exponent zero is equal to 1.

179. Negative Exponents. Assuming Law I to hold also for negative exponents, and letting t be a positive number, integral or fractional, we determine as follows the meaning of b^{-t} (*read b exponent negative t*).

By Law I, $\quad b^t \cdot b^{-t} = b^{t-t} = b^0 = 1.$ § 44

Therefore, $\quad b^{-t} = \dfrac{1}{b^t}.$

Hence a number with a negative exponent means *the same as the reciprocal of the number with a positive exponent of the same absolute value.*

E.g. $a^{-2} = \dfrac{1}{a^2}$; $4^{-\frac{3}{2}} = \dfrac{1}{4^{\frac{3}{2}}} = \dfrac{1}{2^3} = \dfrac{1}{8}.$

ORAL EXERCISES

In the following replace the radicals by fractional exponents:

1. $3\sqrt{2}$.
2. $3\sqrt{x^3y}$.
3. $2\sqrt[3]{x^4}$.
4. $d\sqrt[3]{a^2b^5}$.
5. $\sqrt{a^3}\sqrt{b^{10}}$.
6. $r\sqrt[4]{b^3}$.
7. $\sqrt[4]{b^9} \cdot \sqrt[5]{a^6}$.
8. $c\sqrt[3]{a^5b^2}$.
9. $\sqrt[3]{(a-b)^5}$.

180. New Form of Notation. It thus appears that fractional and negative exponents simply provide *new notations for indicating operations already well known.* Sometimes one notation is more convenient and sometimes the other.

Fractional and negative exponents are also called *powers.*

E.g. $x^{\frac{2}{3}}$ may be read *x to the* $\frac{2}{3}$ *power*, and x^{-4} may be read *x to the negative 4th power.*

Radical signs may now be replaced by fractional exponents, or fractional exponents by radical signs.

E.g. $\sqrt[3]{x^2} + 3\sqrt[5]{x^3} \cdot \sqrt{y} + 5\sqrt[7]{x^4}\sqrt[3]{y^2} \equiv x^{\frac{2}{3}} + 3\,x^{\frac{3}{5}}y^{\frac{1}{2}} + 5\,x^{\frac{4}{7}}y^{\frac{2}{3}}.$

In a fraction, *any factor may be changed from numerator to denominator, or from denominator to numerator by changing the sign of its exponent,* as shown in the following examples:

1. $\dfrac{a^2b^{-3}}{c} = \dfrac{a^2 \cdot \frac{1}{b^3}}{c} = \dfrac{\frac{a^2}{b^3}}{c} = \dfrac{a^2}{b^3c}.$

2. $\dfrac{ab}{x^2} = abx^{-2}$, since $abx^{-2} = ab \cdot \dfrac{1}{x^2} = \dfrac{ab}{x^2}.$

3. $ab^{-3}c^2 = ac^2 \cdot \dfrac{1}{b^3} = \dfrac{ac^2}{b^3}.$

4. $32^{-\frac{4}{5}} = \dfrac{1}{32^{\frac{4}{5}}} = \dfrac{1}{(\sqrt[5]{32})^4} = \dfrac{1}{2^4} = \dfrac{1}{16}.$

WRITTEN EXERCISES

In the following replace fractional exponents by radicals:

1. $a^{\frac{2}{3}}.$
2. $x^{\frac{2}{5}}.$
3. $18^{\frac{1}{2}}.$
4. $a^{\frac{5}{3}}.$
5. $x^{\frac{3}{7}}.$
6. $a^{\frac{7}{5}}.$
7. $m^{\frac{4}{5}}n^{\frac{3}{4}}.$
8. $a^{\frac{3}{4}} \cdot b^{\frac{5}{3}}.$
9. $ax^{\frac{7}{2}}.$
10. $bx^{\frac{3}{4}}.$
11. $a^{\frac{2}{3}}b^{\frac{7}{5}}.$
12. $m^{\frac{3}{4}}n^{\frac{4}{5}}l^{\frac{7}{8}}.$
13. $m^{\frac{3}{2}}n^{\frac{2}{3}}l^{\frac{4}{5}}.$
14. $\dfrac{a^{\frac{2}{3}}b^{\frac{3}{2}}}{a^{\frac{1}{2}}b^{\frac{2}{3}}}.$
15. $\dfrac{4\,a^{\frac{5}{6}} \cdot b^{\frac{3}{2}}}{5\,a^{\frac{1}{2}}b^{\frac{1}{3}}}.$
16. $\dfrac{c^{\frac{1}{2}}d^{\frac{1}{4}}}{d^{\frac{1}{6}}c^{\frac{1}{3}}}.$

In the following change all expressions having negative exponents to equivalent expressions having positive exponents:

17. $\dfrac{1}{b^{-\frac{1}{2}}}$.

18. $a^{-2}b^{3}$.

19. $\dfrac{3}{m^{-7}}$.

20. $\dfrac{3\,a^{-\frac{2}{3}}}{c^{-2}b^{3}}$.

21. $\dfrac{n^{-\frac{1}{3}}m^{-1}}{4}$.

22. $\dfrac{3\,a^{-\frac{2}{3}}b^{-2}}{4\,a^{-\frac{1}{2}}b^{-\frac{1}{2}}}$.

23. $3(a+b)^{-\frac{2}{3}}$.

24. $\left(\dfrac{1}{a}\right)^{-\frac{3}{4}}$.

25. $\left(\dfrac{1}{27}\right)^{-\frac{1}{3}}$.

181. Extended Laws of Exponents. Fractional and negative exponents were defined so as to conform to Law I, §§ 177, 179. It is now possible to show that, when so defined, they also conform to Laws II, III, IV, and V there given. Thus:

Law II. $a^m \div a^n = a^{m-n}$ *for all values of m and n, positive or negative, integral or fractional.*

E.g.

$$a^2 \div a^5 = a^{2-5} = a^{-3}.$$

$$a^{\frac{1}{2}} \div a^{-\frac{1}{3}} = a^{\frac{1}{2}-(-\frac{1}{3})} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{5}{6}}.$$

Law III. $(a^m)^n = a^{mn}$ *for all values of m and n.*

E.g.

$$(a^{-3})^2 = a^{2(-3)} = a^{-6}.$$

$$(a^{\frac{1}{3}})^{-\frac{2}{5}} = a^{\frac{1}{3}(-\frac{2}{5})} = a^{-\frac{2}{15}}.$$

Law IV. $(a^m \cdot b^n)^p = a^{mp} \cdot b^{np}$ *for all values of m, n, and p.*

E.g.

$$(a^{\frac{1}{2}}b^{-3})^{\frac{2}{3}} = a^{\frac{1}{2}\cdot\frac{2}{3}}b^{-3\cdot\frac{2}{3}} = a^{\frac{1}{3}}b^{-2}.$$

$$(a^{-3}b^{2})^{-\frac{1}{6}} = a^{-3(-\frac{1}{6})}b^{2(-\frac{1}{6})} = a^{\frac{1}{2}}b^{-\frac{1}{3}}.$$

Law V. $\left(\dfrac{a^m}{b^n}\right)^p = \dfrac{a^{mp}}{b^{np}}$ *for all values of m, n, and p.*

E.g.

$$\left(\frac{a^{-3}}{b^{\frac{1}{2}}}\right)^{\frac{2}{3}} = \frac{a^{-3\cdot\frac{2}{3}}}{b^{\frac{1}{2}\cdot\frac{2}{3}}} = \frac{a^{-2}}{b^{\frac{1}{3}}}.$$

$$\left(\frac{a^{3}}{b^{-2}}\right)^{-\frac{1}{6}} = \frac{a^{3(-\frac{1}{6})}}{b^{-2(-\frac{1}{6})}} = \frac{a^{-\frac{1}{2}}}{b^{\frac{1}{3}}}.$$

182. Affecting a Monomial with an Exponent. From Laws III, IV, and V, it follows that *any monomial is affected with a given exponent by multiplying the exponent of each factor of the monomial by the given exponent.*

Since exponents may be integral or fractional, positive or negative, the statement just made is equivalent to the two principles given in § 160.

Example 1. $(a^{\frac{1}{2}}b^{-2}c^{3})^{-\frac{2}{3}} = a^{-\frac{2}{3}\cdot\frac{1}{2}}b^{-\frac{2}{3}\cdot(-2)}c^{-\frac{2}{3}\cdot 3} = a^{-\frac{1}{3}}b^{\frac{4}{3}}c^{-2}.$

Example 2. $\left(\dfrac{3\,a^2x^6}{by^4}\right)^{-\frac{1}{2}} = \dfrac{3^{-\frac{1}{2}}a^{-1}x^{-3}}{b^{-\frac{1}{2}}y^{-2}} = \dfrac{b^{\frac{1}{2}}y^2}{3^{\frac{1}{2}}ax^3}.$

Example 3. $\left(\dfrac{8\,x^9}{27\,y^6}\right)^{-\frac{1}{3}} = \left(\dfrac{27\,y^6}{8\,x^9}\right)^{\frac{1}{3}} = \dfrac{27^{\frac{1}{3}}\,y^2}{8^{\frac{1}{3}}\,x^3} = \dfrac{3\,y^2}{2\,x^3}.$

ORAL EXERCISES

Remove the parenthesis in each of the following:

1. $(a^{\frac{1}{3}}b^{\frac{1}{2}}c)^{\frac{1}{2}}$.
2. $(2\,xy^{\frac{2}{3}}z^{\frac{1}{4}})^{2}$.
3. $(3\,a^{\frac{2}{3}}b^{\frac{1}{5}}c^{2})^{3}$.
4. $(8\,x^2y^3)^{-\frac{1}{3}}$.
5. $(16\,a^{-2}b^{-4}c^{6})^{\frac{1}{4}}$.
6. $(32\,a^{-5}b^{10}c)^{\frac{1}{5}}$.
7. $(27\,a^{3}b^{-6}x^{2})^{\frac{1}{3}}$.
8. $(m^{-\frac{3}{2}}n^{\frac{2}{3}}p^{\frac{4}{5}})^{\frac{1}{4}}$.
9. $(r^{\frac{2}{5}}s^{\frac{3}{4}})^{\frac{1}{6}}$.
10. $(x^{-2}y^{3}z^{4})^{\frac{1}{12}}$.
11. $(a^{-\frac{4}{5}}b^{3}c^{-\frac{1}{2}})^{-\frac{1}{2}}$.
12. $(x^{-\frac{1}{3}}y^{-\frac{1}{4}}z^{-\frac{1}{2}})^{-12}$.

WRITTEN EXERCISES

Perform the operations indicated by the exponents in each of the following, writing the results without negative exponents and in as simple form as possible:

1. $(\frac{16}{25})^{-\frac{1}{2}}$.
2. $(\frac{27}{64})^{-\frac{1}{3}}$.
3. $(\frac{25}{36})^{\frac{3}{2}}$.
4. $(27\,a^{-9})^{\frac{1}{3}}$.
5. $(x^{-\frac{2}{3}}y^{\frac{4}{3}})^{-\frac{3}{4}}$.
6. $25^{\frac{3}{2}}$.
7. $25^{-\frac{3}{2}}$.
8. 25^{0}.
9. $(\frac{27}{8})^{-\frac{2}{3}}$.
10. $(\frac{8}{27})^{\frac{2}{3}}$.
11. $(0.25)^{\frac{1}{2}}$.
12. $(0.25)^{-\frac{1}{2}}$.
13. $(.0009)^{\frac{3}{2}}$.
14. $(.027)^{\frac{1}{3}}$.
15. $(32\,a^{-5}b^{10})^{\frac{1}{5}}$.
16. $8^{\frac{1}{3}}\cdot 4^{-\frac{1}{2}}$.

17. $\left(\dfrac{a^{-8}}{16}\right)^{-\frac{1}{4}}$.

18. $(27\,x^6y^{-3}z^{-1})^{-\frac{1}{3}}$.

19. $(\frac{1}{32})^{-\frac{1}{5}}(\frac{1}{81})^{-\frac{3}{4}}$.

20. $\sqrt[3]{\frac{512}{729}}\cdot(\frac{256}{729})^{-\frac{1}{2}}$.

21. $(-\frac{243}{32})^{\frac{1}{5}}\div(\frac{16}{81})^{-\frac{1}{4}}$.

22. $\left(\dfrac{x^3y^{-4}}{x^{-2}y}\right)^3\left(\dfrac{x^{-3}y^2}{xy^{-1}}\right)^5$.

23. $\sqrt[4]{81\,a^{-4}b^8}(-27\,a^3b^{-6})^{-\frac{1}{3}}$.

24. $\left(\dfrac{m^{-1}n}{m^{\frac{1}{2}}n^{-\frac{2}{3}}}\right)^{-2}\div\left(\dfrac{m^{-3}}{n^{-1}}\right)^{-\frac{1}{3}}$.

25. $\left(\dfrac{a^3b^{-2}}{a^{-2}b^3}\right)^{\frac{1}{5}}\div\left(\dfrac{a^3b^{-3}}{a^{-3}b^3}\right)^{-1}$.

26. $\sqrt{16\,a^{-4}b^{-6}}\cdot\sqrt[3]{8\,a^3b^{-6}}$.

27. $(-2^{-2}a^{-3}b^{-6})^{-\frac{1}{3}}(-2^{-\frac{1}{3}}a^{-\frac{1}{2}}b^{-1})^2$.

28. $\left(\dfrac{81\,r^{-16}s^4t}{625\,r^4s^8t}\right)^{\frac{1}{4}}\left(\dfrac{9\,r^2s^{-4}t}{25\,r^{10}s^0t}\right)^{-\frac{1}{2}}$.

Multiply:

29. $x^{-2}+x^{-1}y^{-1}+y^{-2}$ by $x^{-1}-y^{-1}$.

30. $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}$.

31. $x^{\frac{4}{5}}+x^{\frac{3}{5}}y^{\frac{1}{5}}+x^{\frac{2}{5}}y^{\frac{2}{5}}+x^{\frac{1}{5}}y^{\frac{3}{5}}+y^{\frac{4}{5}}$ by $x^{\frac{1}{5}}-y^{\frac{1}{5}}$.

32. $\sqrt[4]{a^3}+\sqrt[5]{b^2}$ by $\sqrt[4]{a^3}-\sqrt[5]{b^2}$.

33. $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}$ by $x^{\frac{2}{3}}-y^{\frac{2}{3}}$.

34. $x-3\,x^{\frac{2}{3}}y^{-\frac{1}{2}}+3\,x^{\frac{1}{3}}y^{-1}-y^{-\frac{3}{2}}$ by $x^{\frac{2}{3}}-2\,x^{\frac{1}{3}}y^{-\frac{1}{2}}+y^{-1}$.

35. $x^{\frac{3}{2}}+xy^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{3}{2}}+y^{\frac{9}{4}}$ by $x^{\frac{1}{2}}-y^{\frac{3}{4}}$.

Divide:

36. $x^2-x^{\frac{11}{6}}y+x^{\frac{3}{2}}y-x^{\frac{1}{2}}y^{\frac{1}{3}}+x^{\frac{1}{3}}y^{\frac{4}{3}}-y^{\frac{4}{3}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{3}}y+y$.

37. $3\,a^{\frac{7}{4}}-ab^{\frac{2}{3}}+4\,ab^2-3\,a^{\frac{3}{4}}b+b^{\frac{5}{3}}-4\,b^3$ by $3\,a^{\frac{3}{4}}-b^{\frac{2}{3}}+4\,b^2$

38. $x^2-3\,x^{\frac{5}{3}}+6\,x^{\frac{4}{3}}-7\,x+6\,x^{\frac{2}{3}}-3\,x^{\frac{1}{3}}+1$ by $x^{\frac{2}{3}}-x^{\frac{1}{3}}+1$.

39. $4\,x^{\frac{5}{4}}b^{-2}-17\,x^{\frac{1}{4}}b^2+16\,x^{-\frac{3}{4}}b^6$ by $2\,x^{\frac{1}{2}}-b^2-4\,x^{-\frac{1}{2}}b^4$.

Find the square root of:

40. $4\,x^2-4\,xy^{\frac{1}{3}}+4\,xz^{-\frac{1}{2}}+y^{\frac{2}{3}}-2\,y^{\frac{1}{3}}z^{-\frac{1}{2}}+z^{-1}$.

41. $a^{-\frac{2}{3}}-2\,a^{-\frac{1}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}+2\,a^{-\frac{1}{3}}c^2+c^4-2\,b^{\frac{2}{3}}c^2$.

42. $b^{-\frac{1}{2}}-2\,b^{-\frac{1}{4}}c^{\frac{1}{3}}+c^{\frac{2}{3}}+2\,b^{-\frac{1}{4}}d^{\frac{1}{5}}+2\,b^{-\frac{1}{4}}e^{-\frac{1}{2}}-2\,c^{\frac{1}{3}}d^{\frac{1}{5}}+d^{\frac{2}{5}}$
$+2\,d^{\frac{1}{5}}e^{-\frac{1}{2}}-2\,c^{\frac{1}{3}}e^{-\frac{1}{2}}+e^{-1}$.

HISTORICAL NOTE

The **full use of exponents**, including fractional and negative, was the culmination of a long historical development. **Bhaskara**, an Indian mathematician of the 11th century, used the initial syllable of the word for *unknown quantity*, repeating this to denote powers. Similar devices prevailed for five hundred years.

For example, in the 16th century, $x^2 + 5x - 4$ would have been written $1\,Z\,p\,5\,R\,m\,4$, where Z stands for *zensus*, i.e. x^2, p for plus, R for *root* (of the *square*), and m for *minus*.

Bombelli (1572) wrote this expression $1\overset{2}{\smile}\,p\,5\overset{1}{\smile}\,m\,4$, where $\overset{1}{\smile}$, $\overset{2}{\smile}$ are used to indicate the first and second powers of the unknown quantity, respectively. Clearly this notation would not permit the introduction of the powers of more than one quantity into any one expression.

Stevinus (1586) used ①, ②, ③, for first, second, and third powers, but this has the same limitation as the notation of Bombelli.

Vieta (1591) wrote A*quad*, A*cub*, ⋯ for A^2, A^3, ⋯ . This had the advantage of permitting the introduction of powers of more than one quantity.

Harriot (1631) wrote aa for a^2, aaa for a^3, etc.

Finally **Descartes** (1637) used the present exponential notation for a^2, a^3, a^4, etc. He did not use fractional or negative exponents.

Wallis (1659) used fractional and negative exponents and explained their meaning in much the same manner as in §§ 177–179 of this book.

Newton (1676) used expressions such as a^n, containing literal exponents, without any restriction as to whether the exponents were fractional or integral, positive or negative.

Euler (1707–1783) showed that logarithms could most naturally be regarded as exponents and that the laws of exponents determine the laws of operations by means of logarithms.

ORAL EXERCISES

1. Read $3x^4 + 2x^3 - 7x + 5$, using Bombelli's notation.

2. In what important respect did Vieta's notation differ from those that preceded him?

3. Read $5x^3 - 7x^2 + 4x + 9$, using Harriot's notation.

4. Who first introduced the notation a^3, etc.?

5. Who first used fractional exponents? When?

REDUCTION OF RADICAL EXPRESSIONS

183. Radical Expression. An expression containing a root indicated by the radical sign or by a fractional exponent is called a *radical expression.* The expression whose root is indicated is the **radicand.**

E.g. $\sqrt[3]{5}$ and $(1 + x)^{\frac{2}{3}}$ are radical expressions. In each case the **index** of the radical is 3.

A reduction of a radical expression consists in *changing its form without changing its value.*

Each reduction is based upon one or more of the Laws I to V, as extended in § 176.

184. Case I. To Remove a Factor from the Radicand. This reduction is possible only when the radicand contains a factor which is a perfect power of the degree indicated by the index of the root, as shown in the following examples:

Example 1. $\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36}\sqrt{2} = 6\sqrt{2}.$ See § 158.

Example 2. $(a^3x^2y^6)^{\frac{1}{3}} = (a^3y^6 \cdot x^2)^{\frac{1}{3}} = (a^3y^6)^{\frac{1}{3}} \cdot (x^2)^{\frac{1}{3}} = ay^2x^{\frac{2}{3}}.$

This reduction involves Law IV, and may be written in symbols thus:

$$\sqrt[r]{x^{kr}y} = \sqrt[r]{x^{kr}}\sqrt[r]{y} = x^k\sqrt[r]{y}.$$

ORAL EXERCISES

In each of the following remove as many factors as possible from the radicand:

1. $\sqrt{45}.$	7. $\sqrt{(a+b)^3}.$	13. $\sqrt[4]{32}.$
2. $\sqrt{125}.$	8. $\sqrt[3]{(a-b)^5}.$	14. $\sqrt[4]{48}.$
3. $\sqrt{50}.$	9. $\sqrt[5]{-a^6b^4c^9}.$	15. $\sqrt[6]{128}.$
4. $\sqrt{72}.$	10. $\sqrt[5]{a^3b^7c^{13}}.$	16. $\sqrt[3]{-a^3b}.$
5. $3\sqrt{x^2y}.$	11. $\sqrt{a^3\sqrt[4]{b^9}}.$	17. $\sqrt[2]{7x^2 - 14xy + 7y^2}.$
6. $2\sqrt[3]{-x^4}.$	12. $\sqrt[3]{b^7}\sqrt[4]{a^5}.$	18. $\sqrt[3]{4a^3 - 12a^2b + 12ab^2 - 4b^3}.$

185. Case II. To Introduce a Factor into the Radicand. This process simply reverses the steps of the foregoing reduction, and hence also involves Law IV.

Example 1. $6\sqrt{2} = \sqrt{6^2} \cdot \sqrt{2} = \sqrt{36 \cdot 2} = \sqrt{72}$. See § 184.

Example 2. $ay^2x^{\frac{2}{3}} = \sqrt[3]{(ay^2)^3} \cdot \sqrt[3]{x^2} = \sqrt[3]{(ay^2)^3x^2} = \sqrt[3]{a^3y^6x^2}$.

Example 3. $x\sqrt[r]{y} = \sqrt[r]{x^r}\sqrt[r]{y} = \sqrt[r]{x^ry}$.

ORAL EXERCISES

In each of the following introduce into the radicand the factor which appears as the coefficient of the radical:

1. $2\sqrt{5}$.
2. $3(7)^{\frac{1}{2}}$.
3. $3\sqrt{xy}$.
4. $d\sqrt[3]{a^2b}$.
5. $c\sqrt[3]{a^5b^2}$.
6. $a\sqrt{x}$.
7. $n\sqrt{m^5}$.
8. $r\sqrt{b^3}$.
9. $3\sqrt[3]{a+b}$.
10. $2\sqrt[5]{cx}$.
11. $2\sqrt[3]{x^4}$.
12. $a^2\sqrt[4]{c^3}$.
13. $a\sqrt[n]{b}$.
14. $a^2\sqrt[n]{a}$.
15. $b\sqrt[n]{b}$.
16. $c^2\sqrt[4]{c}$.
17. $3\sqrt[3]{3}$.
18. $a^4\sqrt[n]{a}$.
19. $x\sqrt[2n]{2}$.
20. $2a\sqrt[n]{a^2}$.
21. $3ab\sqrt[2n]{a^3b^2}$.

186. Case III. To Reduce a Fractional Radicand to the Integral Form. This reduction involves Law IV or Law V, and may always be accomplished.

Example 1. $\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \sqrt{\frac{1}{25} \cdot 15} = \frac{1}{5}\sqrt{15}$. Law IV

Example 2. $\left(\dfrac{a-b}{a+b}\right)^{\frac{1}{2}} = \left(\dfrac{a^2-b^2}{(a+b)^2}\right)^{\frac{1}{2}} = \dfrac{(a^2-b^2)^{\frac{1}{2}}}{[(a+b)^2]^{\frac{1}{2}}} = \dfrac{(a^2-b^2)^{\frac{1}{2}}}{a+b}$. Law V

Example 3. $\dfrac{\sqrt[3]{40}}{\sqrt[3]{5}} = \sqrt[3]{\dfrac{40}{5}} = \sqrt[3]{8} = 2$.

In symbols, we have $\sqrt[r]{\dfrac{a}{b}} = \sqrt[r]{\dfrac{ab^{r-1}}{b^r}} = \dfrac{\sqrt[r]{ab^{r-1}}}{\sqrt[r]{b^r}} = \dfrac{1}{b}\sqrt[r]{ab^{r-1}}$.

ORAL EXERCISES

In each of the following reduce the radicand to the integral form:

1. $\sqrt{\frac{3}{4}}$.	6. $\sqrt[3]{-\frac{1}{4}}$.	11. $\sqrt[3]{\frac{1}{32}}$.	16. $\sqrt{\frac{b}{ac^3}}$.
2. $\sqrt{\frac{5}{9}}$.	7. $\sqrt{\frac{1}{a^3}}$.	12. $\sqrt[3]{-\frac{a}{b}}$.	17. $\sqrt[3]{\frac{a}{b^2c^3}}$.
3. $\sqrt{\frac{1}{2}}$.	8. $\sqrt{\frac{1}{a}}$.	13. $\sqrt{\frac{5}{12}}$.	18. $\sqrt[3]{\frac{ab}{x^2y^4}}$.
4. $\sqrt{\frac{1}{3}}$.	9. $\sqrt[3]{\frac{1}{a^2}}$.	14. $\sqrt{\frac{7}{18}}$.	19. $\sqrt{\frac{1}{27}}$.
5. $\sqrt[3]{\frac{1}{2}}$.	10. $\sqrt{\frac{1}{8}}$.	15. $\sqrt[3]{\frac{a}{xy^2}}$.	20. $\sqrt[3]{-\frac{1}{9}}$.

187. Case IV. To Reduce a Radical to an Equivalent Radical of Lower Index. This reduction can be made when the radicand is a perfect power corresponding to *some factor of the index.*

Example 1. $\sqrt[6]{8}=8^{\frac{1}{6}}=8^{\left(\frac{1}{3\cdot 2}\right)}=(8^{\frac{1}{3}})^{\frac{1}{2}}=2^{\frac{1}{2}}=\sqrt{2}.$

Example 2. $\sqrt[4]{a^2+2ab+b^2}=\sqrt{\sqrt{a^2+2ab+b^2}}=\sqrt{a+b}.$

This reduction involves Law III as follows:

$$(x^{\frac{1}{r}})^{\frac{1}{s}}=(x^{\frac{1}{s}})^{\frac{1}{r}}=x^{\frac{1}{rs}},$$ See § 181

from which we have $$\sqrt[rs]{x}=\sqrt[r]{\sqrt[s]{x}}=\sqrt[s]{\sqrt[r]{x}}.$$

By this reduction the finding of a root whose index is a composite number is made to depend upon roots of lower degree.

E.g. A fourth root may be found by taking the square root twice; a sixth root, by taking a square root and then a cube root, etc.

In some cases the reduction may be made by expressing the power and the root involved by means of a fractional exponent and then reducing this fraction to its lowest terms.

E.g. $$\sqrt[6]{2^3}=2^{\frac{3}{6}}=2^{\frac{1}{2}}=\sqrt{2},$$

and $$\sqrt[4]{a^2+2ab+b^2}=\sqrt[4]{(a+b)^2}=(a+b)^{\frac{2}{4}}=(a+b)^{\frac{1}{2}}.$$

ORAL EXERCISES

Reduce each of the following radicals to an equivalent radical of lower index:

1. $\sqrt[6]{(a-b)^3}$.
2. $\sqrt[4]{a^6}$.
3. $\sqrt[10]{x^5}$.
4. $\sqrt[4]{64}$.
5. $\sqrt[4]{x^2-2xy+y^2}$.
6. $\sqrt[6]{125}$.
7. $\sqrt[6]{(x+y)^3}$.
8. $\sqrt[6]{(x+y)^2}$.
9. $\sqrt[4]{a^2b^6}$.
10. $\sqrt[6]{a^9b^3}$.
11. $\sqrt[4]{225}$.
12. $\sqrt[4]{625\,a^6}$.
13. $\sqrt[4]{(a-b)^6}$.
14. $\sqrt[6]{25\,y^2a^4}$.
15. $\sqrt[8]{16\,x^4y^{12}}$.

188. Case V. To Reduce a Radical to an Equivalent Radical of Higher Index. This reduction is possible whenever the required index is a *multiple* of the given index. Thus:

$$x^{\frac{r}{s}} = \left(x^{\frac{r}{s}}\right)^{\frac{t}{t}} = x^{\frac{rt}{st}}. \qquad \text{Law III. } \S\ 181$$

Example 1. $\sqrt{a} = a^{\frac{1}{2}} = \left(a^{\frac{1}{2}}\right)^{\frac{3}{3}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}$.

Example 2. $\sqrt[3]{b} = b^{\frac{1}{3}} = b^{\frac{2}{6}} = \sqrt[6]{b^2}$.

189. Radicals of the Same Order. Two radical expressions are said to be of the *same order* when their indicated roots have the *same index*.

By case V two radicals of *different* orders may be changed to equivalent radicals of the *same* order. The index of the new radical will be a common multiple of the given indices.

WRITTEN EXERCISES

Reduce each of the following pairs of radicals to radicals of the same order:

1. $x^{\frac{2}{5}}, x^{\frac{1}{3}}$.
2. $3\sqrt{x^2y},\ 2\sqrt[3]{x^4}$.
3. $\sqrt{(a+b)},\ \sqrt[3]{(a-b)^2}$.
4. $\sqrt[4]{3},\ \sqrt[6]{2}$.
5. $\sqrt[9]{3},\ \sqrt[6]{2}$.
6. $ax^{\frac{7}{2}},\ bx^{\frac{3}{4}}$.
7. $4(ab)^{\frac{3}{4}},\ 3(cx)^{\frac{2}{3}}$.
8. $a(xy)^{\frac{2}{3}},\ b(cd)^{\frac{5}{6}}$.

190. Radicals in the Simplest Form. In general, radical expressions should be reduced at once to the lowest possible order and the radicand made *integral and as small* as possible. A radical is then said to be in its **simplest form.**

ADDITION AND SUBTRACTION OF RADICALS

191. Similar Radicals. Two radical expressions are said to be **similar** when they are of the same order and have the same radicands.

E.g. $3\sqrt{7}$ and $5\sqrt{7}$ are similar radicals as are also $a\sqrt[7]{x^4}$ and $b\sqrt[7]{x^4}$.

If two radicals can be reduced to similar radicals, they may be added or subtracted according to § 28.

Example 1. Find the sum of $\sqrt{8}$, $\sqrt{50}$, and $\sqrt{98}$.

By § 184, $\sqrt{8} = 2\sqrt{2}$, $\sqrt{50} = 5\sqrt{2}$, and $\sqrt{98} = 7\sqrt{2}$.
Hence $\sqrt{8} + \sqrt{50} + \sqrt{98} = 2\sqrt{2} + 5\sqrt{2} + 7\sqrt{2} = 14\sqrt{2}$.

Example 2. Simplify $\sqrt{\frac{1}{5}} - \sqrt{20} + \sqrt{3\frac{1}{5}}$.

By § 186, $\sqrt{\frac{1}{5}} = \frac{1}{5}\sqrt{5}$, $\sqrt{20} = 2\sqrt{5}$, $\sqrt{3\frac{1}{5}} = \sqrt{\frac{16}{5}} = 4\sqrt{\frac{1}{5}} = \frac{4}{5}\sqrt{5}$.
Hence $\sqrt{\frac{1}{5}} - \sqrt{20} + \sqrt{3\frac{1}{5}} = \frac{1}{5}\sqrt{5} - 2\sqrt{5} + \frac{4}{5}\sqrt{5} = -\sqrt{5}$.

If two radicals cannot be reduced to equivalent similar radicals, their sum can only be indicated.

E.g. The sum of $\sqrt{2}$ and $\sqrt[3]{5}$ is $\sqrt{2} + \sqrt[3]{5}$.

ORAL EXERCISES

Reduce the following to similar radicals:

1. $\sqrt{\frac{1}{2}}, \sqrt{2}, \sqrt{8}$.
2. $\sqrt{\frac{1}{3}}, \sqrt{3}, \sqrt{12}$.
3. $\sqrt{8}, \sqrt{18}, \sqrt{32}$.
4. $\sqrt{\frac{3}{4}}, \sqrt{12}, \sqrt{27}$.
5. $\sqrt{5}, \sqrt{20}, \sqrt{45}$.
6. $\sqrt{a^3b}, \sqrt{a^3b^3}, \sqrt{ab^5}$.
7. $\sqrt{\frac{1}{8}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{32}}$.
8. $\sqrt{50}, \sqrt{\frac{1}{2}}, \sqrt{72}$.
9. $\sqrt{48}, \sqrt{\frac{1}{3}}, \sqrt{75}$.
10. $\sqrt[3]{\frac{1}{4}}, \sqrt[3]{16}, \sqrt[3]{54}$.

WRITTEN EXERCISES

In the following, reduce to similar radicals and add as indicated:

1. $3\sqrt{45} + 2\sqrt{125}$.
2. $d\sqrt[3]{a^2b^5} + c\sqrt[3]{a^5b^2}$.
3. $a^{\frac{4}{3}} + a^{\frac{7}{3}}$.
4. $n\sqrt{m^5} + m^{\frac{5}{2}}$.
5. $18^{\frac{1}{2}} + \sqrt{32}$.
6. $\sqrt{12} + 48^{\frac{1}{2}}$.
7. $3(50)^{\frac{1}{2}} + 4(72)^{\frac{1}{2}}$.
8. $(40)^{\frac{1}{3}} + (625)^{\frac{1}{3}}$.
9. $\sqrt{28} + 3\sqrt{7} - 2\sqrt{63}$.
10. $\sqrt[3]{24} - \sqrt[3]{81} - \sqrt[3]{\frac{3}{125}}$.
11. $\sqrt[5]{a^6} + \sqrt[5]{a^{11}} - \sqrt[5]{32\,a}$.
12. $2\sqrt{48} - 3\sqrt{12} + 3\sqrt{\frac{1}{3}}$.
13. $\sqrt{\frac{1}{7}} + \sqrt{63} + 5\sqrt{7}$.
14. $\sqrt{99} - 11\sqrt{\frac{1}{11}} + \sqrt{44}$.
15. $2\sqrt{\frac{4}{7}} + 3\sqrt{\frac{9}{7}} + \sqrt{175}$.
16. $\sqrt[4]{\frac{9}{625}} + 6\sqrt{\frac{1}{3}} - \sqrt{12}$.
17. $\sqrt[6]{9} + \sqrt[9]{27} + \sqrt[3]{-24}$.
18. $(x^2 + 1)\sqrt{a^3 + a^2b} - \sqrt{(a^2 - b^2)(a - b)}$.

MULTIPLICATION OF RADICALS

192. Radicals of Same Order. Radicals of the *same order* are multiplied by multiplying the radicands. If they are not of the same order, they may be reduced to the same order according to § 189.

E.g. $\sqrt{a} \cdot \sqrt[3]{b} = a^{\frac{1}{2}}b^{\frac{1}{3}} = a^{\frac{3}{6}}b^{\frac{2}{6}} = \sqrt[6]{a^3} \cdot \sqrt[6]{b^2} = \sqrt[6]{a^3b^2}$.

In many cases this reduction is not desirable. Thus, $\sqrt[3]{x^2} \cdot \sqrt{y^3}$ is written $x^{\frac{2}{3}}y^{\frac{3}{2}}$ rather than $\sqrt[6]{x^4y^9}$.

193. Radicals with Same Base. Radicals with the *same base* are multiplied by first expressing them by means of fractional exponents and then adding the exponents.

E.g. $\sqrt[5]{x^2} \cdot \sqrt{x^3} = x^{\frac{2}{5}} \cdot x^{\frac{3}{2}} = x^{\frac{2}{5}+\frac{3}{2}} = x^{\frac{19}{10}}$.

Again, $\sqrt{2} \cdot \sqrt[3]{2} = 2^{\frac{1}{2}+\frac{1}{3}} = 2^{\frac{5}{6}}$, and $2\sqrt{5} \cdot 3\sqrt{5} = 6 \cdot 5 = 30$.

ORAL EXERCISES

Find the following products:

1. $\sqrt{ab} \cdot \sqrt{a^2b^3}$.
2. $\sqrt[3]{ax} \cdot \sqrt[3]{xy}$.
3. $\sqrt{2} \cdot \sqrt{8}$.
4. $\sqrt{8} \cdot \sqrt{18}$.
5. $\sqrt{3} \cdot \sqrt{27}$.
6. $\sqrt{a} \cdot \sqrt{a^3}$.
7. $\sqrt{a^3b} \cdot \sqrt{ab}$.
8. $\sqrt[3]{a^2b} \cdot \sqrt[3]{ab^3}$.
9. $\sqrt[3]{xy} \cdot \sqrt[3]{x^2y^2}$.
10. $\sqrt[3]{3a} \cdot \sqrt[3]{9a^2}$.
11. $\sqrt{a-b} \cdot \sqrt{a+b}$.
12. $\sqrt{a+3} \cdot \sqrt{a+4}$.
13. $a^{\frac{2}{3}} \cdot a^{\frac{1}{3}}$.
14. $a^{\frac{2}{5}} \cdot a^{\frac{2}{5}}$.
15. $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}$.
16. $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$.
17. $x^{\frac{3}{4}} \cdot x^{\frac{1}{2}}$.
18. $x^{\frac{2}{3}} \cdot x^{\frac{1}{6}}$.

194. The principles just enumerated are used in multiplying polynomials containing radicals.

WRITTEN EXERCISES

Find the following products:

1. $(3+\sqrt{11})(3-\sqrt{11})$.
2. $(3\sqrt{2}+4\sqrt{5})(4\sqrt{2}-5\sqrt{5})$.
3. $(2+\sqrt{3}+\sqrt{5})(3+\sqrt{3}-\sqrt{5})$.
4. $(3\sqrt{2}-2\sqrt{18}+2\sqrt{7})(2\sqrt{2}-\sqrt{18}-\sqrt{7})$.
5. $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})(a^2+ab+b^2)$.
6. $(\sqrt{\sqrt{13}+3})(\sqrt{\sqrt{13}-3})$.
7. $(\sqrt{2+3\sqrt{5}})(\sqrt{2+3\sqrt{5}})$.
8. $(3a-2\sqrt{x})(4a+3\sqrt{x})$.
9. $(3\sqrt{3}+2\sqrt{6}-4\sqrt{8})(3\sqrt{3}-2\sqrt{6}+4\sqrt{8})$.
10. $(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})$.
11. $(a-\sqrt{b}-\sqrt{c})(a+\sqrt{b}+\sqrt{c})$.
12. $(2\sqrt{\tfrac{2}{3}}+3\sqrt{\tfrac{1}{2}}+4\sqrt{\tfrac{3}{2}})(2\sqrt{\tfrac{2}{3}}-5\sqrt{\tfrac{1}{2}})$.
13. $(\sqrt[3]{a^2}+\sqrt[3]{b^2})(\sqrt[6]{a^2}+\sqrt[3]{a^2b^2}+\sqrt[9]{b^3})$.
14. $(\sqrt[4]{x^3}-y^3)^3$.

DIVISION OF RADICALS

195. Radicals are divided in accordance with Laws II and V. That is, the exponents are *subtracted* when the *bases* are the *same*, and the bases are *divided* when the *exponents* are the *same*. See § 181.

Example 1. $\sqrt[5]{x^2} \div \sqrt{x^3} = x^{\frac{2}{5}} \div x^{\frac{3}{2}} = x^{\frac{2}{5}-\frac{3}{2}} = x^{-\frac{11}{10}}.$

Example 2. $x^{\frac{2}{3}} \div y^{\frac{2}{3}} = \left(\frac{x}{y}\right)^{\frac{2}{3}} = (xy^{-1})^{\frac{2}{3}} = \sqrt[3]{x^2y^{-2}}.$

Example 3. $\sqrt{a} \div \sqrt[3]{b} = a^{\frac{3}{6}} \div b^{\frac{2}{6}} = \left(\frac{a^3}{b^2}\right)^{\frac{1}{6}} = \sqrt[6]{a^3b^{-2}}.$

ORAL EXERCISES

Find the following quotients:

1. $\sqrt{12} \div \sqrt{3}.$
2. $-\sqrt{8} \div \sqrt{2}.$
3. $-\sqrt{18} \div \sqrt{2}.$
4. $\sqrt{27} \div \sqrt{3}.$
5. $\sqrt{15} \div (-\sqrt{5}).$
6. $\sqrt{a^3} \div \sqrt{a}.$
7. $\sqrt{a^3b} \div \sqrt{ab}.$
8. $\sqrt[3]{a^2x} \div \sqrt[3]{ax}.$
9. $\sqrt[3]{(a+b)^2} \div \sqrt[3]{a+b}.$
10. $\sqrt[3]{x^2y^2} \div \sqrt[3]{xy}$
11. $\sqrt{32} \div (-\sqrt{8}).$
12. $\sqrt[3]{32} \div \sqrt[3]{4}.$
13. $a^{\frac{1}{2}} \div a$
14. $a^{\frac{1}{3}} \div a^{\frac{1}{6}}.$
15. $x^{\frac{1}{2}} \div x^{\frac{1}{4}}.$
16. $x^{\frac{5}{6}} \div x^{\frac{1}{3}}.$
17. $x^{\frac{2}{3}} \div x^{\frac{1}{4}}.$
18. $x^{\frac{4}{5}} \div x^{\frac{7}{10}}.$

WRITTEN EXERCISES

Perform the following divisions:

1. $(\sqrt{a^3} + 2\sqrt{a^5} - 3\sqrt{a}) \div 6\sqrt{a}.$
2. $(\sqrt{a} + \sqrt[3]{b} - c) \div \sqrt{c}.$
3. $(2\sqrt[3]{9} + 3\sqrt[3]{12} - 4\sqrt[3]{15}) \div \sqrt[3]{3}.$
4. $(4\sqrt[5]{7} - 8\sqrt[5]{21} + 6\sqrt[5]{42}) \div 2\sqrt[5]{7}.$
5. $(\sqrt[3]{x^4} - 2\sqrt[3]{x^5} + 3\sqrt[3]{x^8}) \div \sqrt[3]{x}.$
6. $(\sqrt[4]{x^7} + 7\sqrt[4]{x^5} + 2\sqrt[4]{x^3}) \div \sqrt[4]{x^3}.$

196. Rationalizing the Denominator. In case division by a radical expression cannot be carried out conveniently as in the foregoing examples, we indicate the division in the fractional form and then *rationalize* the denominator when possible.

Example 1. $\dfrac{\sqrt{2}}{\sqrt{5}}=\dfrac{\sqrt{2}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}}=\dfrac{\sqrt{10}}{5}.$

Example 2. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}=\dfrac{\sqrt{a}(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}=\dfrac{a+\sqrt{ab}}{a-b}.$

Evidently, this reduction is always possible when the divisor is a *monomial* or *binomial* radical expression of the second order.

The expression by which numerator and denominator are multiplied is called the *rationalizing factor.*

For a monomial denominator, $\sqrt{x}$, the rationalizing factor is $\sqrt{x}$ itself. If the denominator is $\sqrt{x}+\sqrt{y}$ the rationalizing factor is $\sqrt{x}-\sqrt{y}$ and if the denominator is $\sqrt{x}-\sqrt{y}$ the rationalizing factor is $\sqrt{x}+\sqrt{y}$.

WRITTEN EXERCISES

Reduce each of the following to equivalent fractions having a rational denominator:

1. $\dfrac{3}{2-\sqrt{5}}.$

2. $\dfrac{7}{\sqrt{5}+\sqrt{3}}.$

3. $\dfrac{\sqrt{27}}{\sqrt{3}+\sqrt{11}}.$

4. $\dfrac{2-\sqrt{7}}{2+\sqrt{7}}.$

5. $\dfrac{3\sqrt{2}-2\sqrt{3}}{\sqrt{2}+\sqrt{3}}.$

6. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}.$

7. $\dfrac{3\sqrt{3}-2\sqrt{2}}{3\sqrt{3}+2\sqrt{2}}.$

8. $\dfrac{\sqrt{a^2+1}-\sqrt{a^2-1}}{\sqrt{a^2-1}+\sqrt{a^2+1}}.$

9. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}.$

10. $\dfrac{\sqrt{a-b}-\sqrt{a+b}}{\sqrt{a-b}+\sqrt{a+b}}.$

197. In **approximating the value** of an expression such as $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$, reduce it to the form $\frac{10+2\sqrt{21}}{4}$. Note the resulting economy of work.

WRITTEN EXERCISES

Reduce each of the following to the simplest form for approximating the numerical value, and find the results.

1. $\frac{3\sqrt{5}+4\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.
2. $\frac{\sqrt{7}}{\sqrt{7}-\sqrt{2}}$.
3. $\frac{4\sqrt{3}}{\sqrt{3}-\sqrt{2}}$.
4. $\frac{11\sqrt{5}-3\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$.
5. $\frac{7\sqrt{5}+3\sqrt{8}}{2\sqrt{5}-3\sqrt{2}}$.
6. $\frac{5\sqrt{19}-3\sqrt{7}}{3\sqrt{7}-\sqrt{19}}$.
7. $\frac{3\sqrt{2}-\sqrt{5}}{\sqrt{5}-6\sqrt{2}}$.
8. $\frac{5\sqrt{6}-7\sqrt{13}}{3\sqrt{13}-7\sqrt{6}}$.
9. $\frac{2\sqrt{3}+3\sqrt{2}}{6\sqrt{10}-5\sqrt{15}}$.

QUADRATIC SURDS

198. A **surd** is an indicated root of a rational number, which is not reducible to a rational number.

E.g. $\sqrt{2}$ is a surd since it cannot be reduced to a rational number. $\sqrt[3]{4}$, $\sqrt[5]{3}$ are surds for the same reason. $\sqrt{9}$ is not a surd since $\sqrt{9}=3$. $\sqrt{2+\sqrt{2}}$ is not a surd, since $2+\sqrt{2}$ is not a rational number.

199. Order of a Surd. The *order of a surd* is indicated by the **index** of the root.

E.g. $\sqrt[3]{4}$ is a surd of the *third order*, or of index three; $\sqrt[5]{3}$ is a surd of the fifth order or of index five.

200. Quadratic Surd. Surd expressions containing no indicated roots except square roots are called **quadratic surds.**

E.g. $\sqrt{7}$, $\sqrt{2}+\sqrt{3}$, $3+\sqrt{5}$, $\frac{3}{\sqrt{7}-\sqrt{5}}$, are quadratic surds.

201. Square Root of a Binomial Quadratic Surd. A binomial quadratic surd is sometimes a perfect square, and its square root may be found by inspection.

In order to do this we square the expression $\sqrt{a}+\sqrt{b}$ and examine the result, thus:

$$(\sqrt{a}+\sqrt{b})^2=(\sqrt{a})^2+2\sqrt{a}\sqrt{b}+(\sqrt{b})^2=a+2\sqrt{ab}+b=a+b+2\sqrt{ab}.$$

Hence, if in a quadratic surd of the form $x+2\sqrt{y}$, x is the *sum of two numbers, a and b, and y is the product of these numbers, then $\sqrt{a}+\sqrt{b}$ is the square root of $x+2\sqrt{y}$.*

Example 1. Find the square root of $5+\sqrt{24}$.

Solution. First put this in the *type form* $x+2\sqrt{y}$, thus,

$$5+\sqrt{24}=5+\sqrt{4\cdot 6}=5+2\sqrt{6}.$$

We now seek two numbers whose sum is 5 and whose product is 6. There are two integers, namely 3 and 2, which fulfill these conditions.

Hence, $\sqrt{5+\sqrt{24}}=\sqrt{3}+\sqrt{2}.$

Check. $(\sqrt{3}+\sqrt{2})^2=(\sqrt{3})^2+2\sqrt{3\cdot 2}+(\sqrt{2})^2=3+2+2\sqrt{3\cdot 2}$
$=5+2\sqrt{6}.$

Example 2. Find the square root of $8-\sqrt{60}$.

Solution. $8-\sqrt{60}=8-2\sqrt{15}$. The two integers whose sum is 8 and whose product is 15 are 5 and 3.

Hence, $\sqrt{8-2\sqrt{15}}=\sqrt{5}-\sqrt{3}.$

Check. $(\sqrt{5}-\sqrt{3})^2=5+3-2\sqrt{15}=8-\sqrt{60}.$

WRITTEN EXERCISES

Find the square root of each of the following:

1. $3-2\sqrt{2}$.
2. $7+\sqrt{40}$.
3. $8-\sqrt{60}$.
4. $7+4\sqrt{3}$.
5. $24-6\sqrt{7}$.
6. $28+3\sqrt{12}$.
7. $9-6\sqrt{2}$.
8. $10-4\sqrt{6}$.
9. $9+4\sqrt{5}$.
10. $9+2\sqrt{14}$.
11. $12-2\sqrt{35}$.
12. $12-8\sqrt{2}$.
13. $11+6\sqrt{2}$.
14. $11-4\sqrt{6}$.
15. $11+4\sqrt{7}$.
16. $11-2\sqrt{30}$.
17. $10+2\sqrt{21}$.
18. $13-2\sqrt{22}$.

PROBLEMS INVOLVING RADICALS

202. By means of **radicals and roots** we are now able to solve equations of the type $x^2 = a$.

Example 1. Solve the equation $x^2 = 32$.

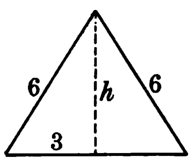
Solution. Taking square roots, $x = \pm\sqrt{32} = \pm 4\sqrt{2}$.
If it is desired to express this root in decimals, we approximate the square root of 2 and multiply the result by 4.

Example 2. Solve $x^2 = 6 + 2\sqrt{5}$.

Solution. By § 201, $x = \sqrt{6 + 2\sqrt{5}} = \pm(\sqrt{5} + 1)$.

In case such an equation arises in a concrete problem, we must decide by direct application to the problem which of the two roots gives the required solution.

Example 3. Find the altitude of an equilateral triangle whose sides are each 6.



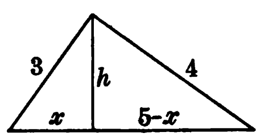
Solution. By the right triangle proposition

$$h^2 = 6^2 - 3^2 = 36 - 9 = 27. \quad \text{(See the figure.)}$$

Hence, $h = \pm\sqrt{27} = \pm 3\sqrt{3}$.

Clearly the altitude of a triangle cannot be negative.
Hence, the altitude is $h = 3\sqrt{3}$.

Example 4. Find the altitude h on the longest side of a triangle whose sides are 3, 4, 5.



Solution. Using the notation of the figure,

$h^2 = 9 - x^2$ and also $h^2 = 16 - (5 - x)^2$.

Hence, equating the two values of h^2,

$$9 - x^2 = 16 - (5 - x)^2,$$

or

$$9 - \cancel{x^2} = 16 - 25 + 10x - \cancel{x^2}.$$

Hence, $x = 1\frac{4}{5}$.

Substituting this value of x in $h^2 = 9 - x^2$,

we have $h^2 = 9 - (\frac{9}{5})^2 = 9 - \frac{81}{25} = \frac{144}{25}$.

Hence, $h = \pm\sqrt{\frac{144}{25}} = \pm\frac{12}{5}$.

The negative result is not applicable, and hence the altitude is $h = \frac{12}{5}$.

PROBLEMS INVOLVING RADICALS

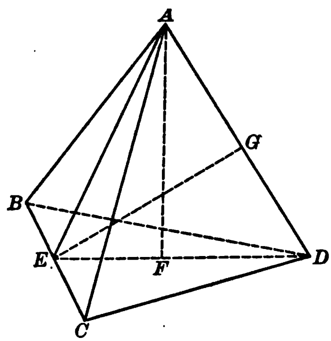
1. Find the area of an equilateral triangle whose sides are 10.

2. Find the area of an equilateral triangle whose sides are a.

3. Find the area of a triangle whose sides are 6, 8, and 10.

4. Find the area of a triangle whose base is a and whose two other sides are a and b.

A three-sided pyramid all of whose edges are equal is called a regular tetrahedron. In the figure AB, AC, AD, BC, BD, CD are all equal.



5. Find the altitude of a regular tetrahedron whose edges are each 6. Also the area of the base.

Hint. First find the altitudes AE and DE, and then find the altitude of the triangle AED on the side DE, *i.e.* find AF. Use Example 4 above.

6. Find the volume of a regular tetrahedron whose edges are each 6. Also, find the volume if the edges are 10.

The volume of a tetrahedron is $\frac{1}{3}$ the product of the base and the altitude.

7. Find the volume of a regular tetrahedron whose edges are each a.

8. In the figure above find EG if the edges are each a.

9. If in the figure EG is 12, compute the volume.

Use problem 8 to find the edge, then use problem 7 to find the volume.

10. Express the volume of the tetrahedron in terms of EG. That is, if $EG = b$, find a general expression for the volume in terms of b.

11. If the altitude of a regular tetrahedron is 10, compute the edge accurately to two places of decimals.

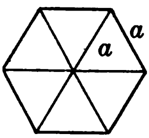
12. Express the edge of a regular tetrahedron in terms of its altitude.

13. Express the volume of a regular tetrahedron in terms of its altitude.

14. Express the edge of a regular tetrahedron in terms of its volume.

15. Express the altitude of a regular tetrahedron in terms of its volume.

16. Express EG of the above figure in terms of the volume of the tetrahedron.



17. Find the area of a regular hexagon whose sides are a.

Suggestion. The regular hexagon may be divided into six equilateral triangles whose sides are a.

18. Find the length of a side of an equilateral triangle whose area is 25. See Example 2 above.

19. Find the length of a side of an equilateral triangle whose area is A.

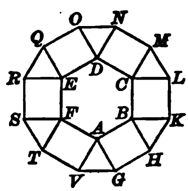
Solve Example 18 by substituting in the formula thus obtained.

20. Find the length of a side of a regular hexagon whose area is 50.

Suggestion. First find the area of a regular hexagon whose sides are a.

21. Find the length of a side of a regular hexagon whose area is A.

Solve Example 20 by means of the formula thus obtained.



22. Find the area of a regular dodecagon whose sides are a. Express the result in simplest form.

A regular dodecagon (twelve-sided figure) is made up of a regular hexagon, six squares and six equilateral triangles all having equal sides.

23. Find the area of a regular dodecagon whose sides are 10.

24. Find the length of a side of a regular dodecagon whose area is A. Express the result in simplest form.

25. Find the volume of a pyramid whose altitude is 7 and whose base is a regular hexagon whose sides are 7.

The volume of a pyramid or of a cone is $\frac{1}{3}$ the product of its base and its altitude.

26. If the volume of the pyramid in problem 25 were 100 cubic inches, what would be its altitude, a side of the base and the altitude being equal? Approximate the result to two places of decimals.

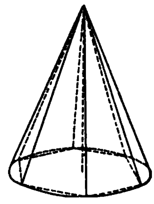
27. If in a right prism the altitude is equal to a side of the base, find the volume, the base being an equilateral triangle whose sides are a.

The volume of a right prism or cylinder equals the product of its base and its altitude.

28. Find the volume of the prism in problem 27, if its base is a regular hexagon whose side is a.

29. Express the side of the base of the prism in problem 27 in terms of its volume. State and solve a particular problem by means of the formula thus obtained.

30. Express the side of the base of the prism in problem 28 in terms of its volume. State and solve a particular problem by means of the formula thus obtained.



31. In the accompanying figure the altitude is supposed to be three times the side a of the regular hexagonal base. Express the difference between the volume of the pyramid and that of its circumscribed cone in terms of a.

The volume of a cone equals $\frac{1}{3}$ the product of its base and altitude.

CHAPTER XIII

THE NUMBER SYSTEM OF ALGEBRA

203. Growth of the Number System. The number system of Algebra is the result of a *gradual growth* starting with the integers of Arithmetic. Each step in this growth was due largely to a desire to make the operations of addition, subtraction, multiplication, and division possible without any exceptions.

204. Operations on the Positive Integers. Starting with the integers of Arithmetic, that is, with the positive integers, the operations of addition and multiplication are always possible in the sense that the result is again a positive integer.

E.g. $2 + 5 = 7$; $3 \cdot 8 = 24$.

However, division of one positive integer by another does not always lead to another integer, and in order to make this operation possible without exception, *we need the positive fractions.*

E.g. $8 \div 4 = 2$; but $2 \div 3 = \frac{2}{3}$.

Again, subtraction of one positive number from another does not always lead to a positive number, and in order to make this operation possible without exception, *we need the negative numbers and zero.*

E.g. $5 - 8 = -3$; and $5 - 5 = 0$.

205. Possible Operations. With the number system enlarged to include *fractions*, *negative numbers*, and *zero*, the operations of addition, subtraction, multiplication, and division are always possible, with one single exception, namely, *it is not possible to divide by zero.* See § 31.

206. Division and Subtraction always possible in Extended Systems. We therefore say that the fractions are brought into the number system to make division possible in all cases (except division by zero); and that the negative numbers and zero are brought in to make subtraction possible without exception.

These statements may be expressed as follows:

We require the number system to be such that the equation $ax + b = 0$ shall have a solution for all values of a and b.

207. Further Extensions of the Number System. The operation of finding roots is *not possible* in all cases, unless other numbers besides positive and negative integers and fractions are admitted to the number system.

E.g. The number $\sqrt{2}$ is not an integer since $1^2 = 1$ and $2^2 = 4$. Suppose $\sqrt{2} = \frac{a}{b}$, a fraction reduced to its lowest terms, so that a and b have no common factor. Then $\frac{a^2}{b^2} = 2$. But this is impossible, for if b^2 is exactly divisible by a^2, then a and b must have factors in common. Hence, $\sqrt{2}$ is not a fraction.

Therefore, if the operation of finding roots is to be possible without exception, a further addition must be made to the number system as explained below.

208. If a positive number is not the square of either an integer or a fraction, a number may be found in terms of integers and fractions whose square differs from the given number by as little as we please.

E.g. 1.41, 1.414, 1.4142, are successive numbers whose squares differ by less and less from 2. In fact $(1.4142)^2$ differs from 2 by less than .0004, and by continuing the process by which these numbers are found, a number may be reached whose square differs from 2 by as little as we please.

1., 1.4, 1.41, 1.414, 1.4142, etc., are successive approximations to the number which we call the square root of 2, and which we represent by the symbol $\sqrt{2}$.

209. The Irrational Number. If a number is neither an integer nor a fraction, but if it can be approximated by means of integers and fractions to any specified degree of accuracy, then such a number is called an *irrational number.*

E.g. $\sqrt{2}$, $3\sqrt{2}$, $5\sqrt[3]{2}$, etc., are irrational numbers, whereas $\sqrt{4}$, $\sqrt[3]{8}$ are rational numbers.

The Number π. It will be found in higher work that there are other irrational numbers besides indicated roots. For instance, the number π, which is the ratio of the circumference of a circle to its diameter, is an irrational number, though it is not an indicated root.

It is shown in higher algebra that irrational numbers correspond to definite points on the line of the number scale, just as do integers and fractions.

210. The Real Number System. We, therefore, enlarge the number system so as to include *irrational numbers* as well as integers and fractions. The set of numbers consisting of all rational and irrational numbers is called the *real number system.*

Certain irrational numbers are brought into the number system by the requirement that *every positive number shall have at least one nth root.* This requirement may also be expressed by saying that the number system shall be such that every equation of the form $x^n = a$ shall have at least one solution. Numbers like π are not brought in by this requirement.

211. Imaginary Numbers. Even with the number system as thus enlarged, it is not possible to find roots in all cases. The exception is the *even root of a negative number,* which is called an *imaginary number.*

E.g. $\sqrt{-4}$ is neither $+2$ nor -2, since $(+2)^2 = +4$, and $(-2)^2 = +4$ and no approximation to this root can be found as in the case of $\sqrt{2}$.

The Imaginary Unit. The expression $\sqrt{-1}$ is defined by the equation $(\sqrt{-1})^2 = -1$, and is usually represented by the letter i. See § 150.

212. Type Form of Complex Numbers. Every expression which can be reduced to the form $a+bi$, in which a and b are real numbers, is called a *complex number.* a is called the *real part,* and bi, the *imaginary part* of the complex number $a+bi$.

The form $a+bi$ is the **type form** of the complex number.

In $a+bi$, if $a=0$, then bi is called a *pure imaginary.* If $b=0$, then a is simply a *real number.*

The complex numbers are brought into the number system by the requirement that every equation of the form $ax^2+bx+c=0$ shall have a solution.

213. The Complete Number System of Algebra. The set of numbers consisting of the *integers* and *fractions* of arithmetic, together with the *negative numbers,* the *irrational numbers,* and the *complex numbers,* is called the *number system of algebra.*

With the number system as thus enlarged, it can be shown that all the algebraic operations, namely, addition, subtraction, multiplication, division, raising to powers, and extracting roots, when applied to numbers in the system lead again to numbers in the system, *the only exception being division by zero.*

In this sense the number system of algebra is *complete in itself.*

Example 1. Reduce $4+\sqrt{-36}$ to the type form $a+bi$.

Solution. $4+\sqrt{-36}=4+\sqrt{36(-1)}=4+6\sqrt{-1}=4+6i$.

Example 2. Reduce $6+\sqrt{-14}$ to type form.

Solution. $6+\sqrt{-14}=6+\sqrt{14\cdot(-1)}=6+\sqrt{14}\sqrt{-1}=6+\sqrt{14}\,i$.

ORAL EXERCISES

Reduce each of the following to type form:

1. $\sqrt{-16}+7$.	4. $\sqrt{-6}+\sqrt{8}$.	7. $5-\sqrt{-16}$.
2. $\sqrt{-25}+3$.	5. $\sqrt{-2}+13$.	8. $6-\sqrt{-10}$.
3. $\sqrt{12}+\sqrt{-12}$.	6. $\sqrt{32}+\sqrt{-4}$.	9. $7-\sqrt{-18}$.

HISTORICAL NOTE

Number. "The conception of 'number' has been much extended in recent times. With the Greeks it included the ordinary positive whole numbers; **Diophantus** (died about 330 A.D.) added rational fractions to the domain of numbers. Later negative numbers and imaginaries came gradually to be recognized.

"**Descartes** fully grasped the notion of the negative; **Gauss**, that of the imaginary. With **Euclid**, a ratio, whether rational or irrational, was not a number. The recognition of ratios and irrationals as numbers took place in the sixteenth century, and found expression with **Newton**.

"By the ratio method, the continuity of the real number system has been based on the continuity of space, but in recent times three theories of irrationals have been advanced by **Weiertrass**, T. W. R. **Dedekind**, G. **Cantor**, and **Heine**, which prove the continuity of numbers without borrowing it from space. They are based on the definition of numbers by regular sequences, the use of series and limits, and some new mathematical conceptions." — CAJORI, *A History of Mathematics*.

That there *are* numbers which are not roots of rational equations became known early in the 19th century. One of the first definite numbers found to be such was the number π (1882).

ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

214. Complex numbers are added by combining the real parts and the imaginary parts separately.

Thus,

$$\begin{array}{r} a + bi \\ c + di \\ \hline a + c + (b + d)i \end{array}$$

The real part of the sum is $a + c$, and the imaginary part is $(b + d)i$.

Similarly, $(a + bi) - (c + di) = (a - c) + (b - d)i$.

Example 1. Simplify $(2 + \sqrt{-4}) + (3 + \sqrt{-9}) + (\sqrt{-16})$.

Solution. $2 + \sqrt{-4} = 2 + 2\sqrt{-1}$; $3 + \sqrt{-9} = 3 + 3\sqrt{-1}$; and $\sqrt{-16} = 4\sqrt{-1}$.

Hence, adding the real parts separately, and the imaginary parts separately, the sum is $(2 + 3) + (2 + 3 + 4)(\sqrt{-1}) = 5 + 9\sqrt{-1} = 5 + 9i$.

Example 2. Simplify $(1+\sqrt{-5})+(5+\sqrt{-3})$
$+(2-\sqrt{-10})$.

Solution: $1+\sqrt{-5}=1+\sqrt{5}\cdot\sqrt{-1}$; $5+\sqrt{-3}=5+\sqrt{3}\cdot\sqrt{-1}$; $2-\sqrt{-10}=2-\sqrt{10}\cdot\sqrt{-1}$.

Adding the real and the imaginary parts separately, the sum is

$$(1+5+2)+(\sqrt{5}+\sqrt{3}-\sqrt{10})\sqrt{-1}=8+(\sqrt{5}+\sqrt{3}-\sqrt{10})i.$$

WRITTEN EXERCISES

Simplify the following:

1. $2\sqrt{-8}$.
2. $\sqrt{-4x^2}+3\sqrt{-9x^2}$.
3. $\sqrt{-8}+\sqrt{-32}$.
4. $3\sqrt{-1}+\sqrt{-9}$.
5. $7\sqrt{-4}-2\sqrt{-9}$.
6. $\sqrt{-4a^2}+\sqrt{-9b^2}$.
7. $3+2\sqrt{-16}+5-\sqrt{-36}$.
8. $\sqrt{-1}+3\sqrt{-4}+\sqrt{-9}$.
9. $\sqrt{-4}-\sqrt{-1}+\sqrt{-16}$.
10. $\sqrt{-9}-\sqrt{-1}+\sqrt{-25}$.
11. $\sqrt{-64}+7\sqrt{-1}-3\sqrt{-4}$.
12. $\sqrt{-81}+\sqrt{-64}-\sqrt{-100}$.
13. $3\sqrt{-4x^4}-\sqrt{-9x^4}+\sqrt{-16x^4}$.
14. $\sqrt{-a^2-2a-1}+\sqrt{-a^2}-\sqrt{-1}$.
15. $\sqrt{-4x^2-4x-1}-\sqrt{-4x^2}+\sqrt{-4}$.
16. $3x\sqrt{-x^2}-3x^2\sqrt{-1}$.
17. $5x\sqrt{-9x^2}+3x^2\sqrt{-4}$.
18. $4a^2\sqrt{-16a^2}+2a^3\sqrt{-25}$.
19. $7\sqrt{-4a^4}+2a\sqrt{-a^2}+4a^2\sqrt{-4}$.
20. $3mn\sqrt{-1}+4m\sqrt{-n^2}-n\sqrt{-m^2}$.
21. $ab\cdot i+2a\sqrt{-b^2}+3b\sqrt{-a^2}$.
22. $x\sqrt{-y^2}+y\sqrt{-x^2}+xy\cdot i$.
23. $4a\sqrt{-b^2c^4}+c^2\sqrt{-a^2b^2}+bc^2\sqrt{-a^2}$.

MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS

215. Multiplication and Division of Pure Imaginary Numbers. By the definition of the imaginary unit, $(\sqrt{-1})^2 = -1$.

This forms the basis for the laws that govern the multiplication and division of pure imaginaries.

Examples:

$$\sqrt{-4}\cdot\sqrt{-9} = 2\sqrt{-1}\cdot 3\sqrt{-1} = 6(\sqrt{-1})^2 = -6.$$

and in general,

$$\sqrt{-a}\cdot\sqrt{-b} = \sqrt{a}\cdot\sqrt{-1}\cdot\sqrt{b}\cdot\sqrt{-1} = \sqrt{ab}(\sqrt{-1})^2 = -\sqrt{ab}.$$

Again, $$\frac{\sqrt{-4}}{\sqrt{-9}} = \frac{2\sqrt{-1}}{3\sqrt{-1}} = \frac{2}{3},$$

and in general, $$\frac{\sqrt{-a}}{\sqrt{-b}} = \frac{\sqrt{a}\cdot\sqrt{-1}}{\sqrt{b}\cdot\sqrt{-1}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Also, $$\frac{\sqrt{-a}}{\sqrt{b}} = \frac{\sqrt{a}\cdot\sqrt{-1}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}}\cdot\sqrt{-1} = \frac{\sqrt{a}}{\sqrt{b}}i.$$

216. Multiplication of Complex Numbers. Two complex numbers of the form $a + bi$ and $c + di$ are multiplied like two binomials, remembering that $(i)^2 = -1$.

Thus,
$$\begin{array}{l} a + bi \\ \underline{c + di} \\ ac + bci \\ \qquad\underline{adi + bd(i)^2} \\ ac + (bc + ad)i - bd = ac - bd + (bc + ad)i. \end{array}$$

217. The Powers of i. The powers of the imaginary unit, $\sqrt{-1} = i$, are of special interest. These are based upon the definition, $(\sqrt{-1})^2 = -1$, or $i^2 = -1$.

Hence we have

$$\left\{\begin{array}{l} i^1 = i. \\ i^2 = -1. \\ i^3 = i^2\cdot i = (-1)i = -i. \\ i^4 = i^2\cdot i^2 = (-1)(-1) = 1. \end{array}\right. \qquad \left\{\begin{array}{l} i^5 = i^4\cdot i = 1\cdot i = i. \\ i^6 = i^4\cdot i^2 = 1(-1) = -1. \\ i^7 = i^4\cdot i^3 = 1(-i) = -i. \\ i^8 = i^4\cdot i^4 = 1\cdot 1 = 1. \end{array}\right.$$

218. Division by a Complex Number. The division of $a + bi$ by $c + di$ may be indicated in the fractional form:

$$\frac{a+bi}{c+di}.$$

Multiplying both terms of the fraction by $c - di$, we have

$$\frac{(c-di)(a+bi)}{(c-di)(c+di)} = \frac{ac+bci-adi+bd}{c^2-(di)^2} = \frac{ac+bd+(bc-ad)i}{c^2+d^2},$$

in which the denominator is a real number. The fraction may be written

$$\frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

which is in the form $A + Bi$, where $A = \frac{ac+bd}{c^2+d^2}$ and $B = \frac{bc-ad}{c^2+d^2}$.

Hence the quotient $(a + bi) \div (c + di)$ has been reduced to the standard form of a complex number.

Note that $$\frac{\sqrt{a}}{\sqrt{-b}} = \frac{\sqrt{a}}{\sqrt{b}\cdot\sqrt{-1}} = \frac{\sqrt{a}\sqrt{-1}}{\sqrt{b}(\sqrt{-1})^2} = -\frac{\sqrt{a}}{\sqrt{b}}\,i.$$

ORAL EXERCISES

Simplify the following:

1. $\sqrt{-4}\cdot\sqrt{-9}$.
2. $\sqrt{-4}\cdot\sqrt{-16}$.
3. $\sqrt{-9}\cdot\sqrt{-25}$.
4. $\sqrt{-16}\cdot\sqrt{-36}$.
5. $\sqrt{-9}\div\sqrt{-4}$.
6. $\sqrt{-16}\div\sqrt{-4}$.
7. $\sqrt{-25}\div\sqrt{-9}$.
8. $\sqrt{-36}\div\sqrt{-36}$.
9. $\sqrt{-100}\div\sqrt{-25}$.
10. $\sqrt{-81}\div\sqrt{-9}$.
11. $i\cdot i = i^2 = -1$.
12. $i\cdot i\cdot i = i^3 = i^2\cdot i = -i$.
13. $-i^5$.
14. i^7.
15. i^9.
16. i^{10}.
17. i^{11}.
18. i^{12}.
19. i^{15}.
20. i^{17}.
21. $-i^{21}$.
22. i^{14}.
23. $-i^{16}$.
24. i^{18}.
25. i^{20}.
26. i^{22}.
27. i^{26}.

WRITTEN EXERCISES

Find the following products:

1. $\sqrt{-3} \cdot \sqrt{-12}$.
2. $-\sqrt{-2} \cdot \sqrt{-8}$.
3. $-\sqrt{-5} \cdot \sqrt{-125}$.
4. $4\sqrt{-18} \cdot \sqrt{-2}$.
5. $\sqrt{-x^2} \cdot \sqrt{-x^2}$.
6. $\sqrt{-6} \cdot \sqrt{-18} \cdot \sqrt{-3}$.
7. $\sqrt{-2} \cdot \sqrt{-32} \cdot (\sqrt{-1})^2$.
8. $3\sqrt{-3} \cdot (-\sqrt{-27})$.
9. $2\sqrt{-2} \cdot 4\sqrt{-3}$.
10. $2\sqrt{-32} \cdot \sqrt{-8}$.
11. $(2+3i)(1-2i)$.
12. $(1-4i)(4-i)$.
13. $(\sqrt{2}+2i)(\sqrt{8}-4i)$.
14. $(4-3i)(9+2i)$.
15. $(7+2i)(2-7i)$.
16. $(x+yi)(x-yi)$.
17. $(x+yi)(x+yi)$.
18. $(x-yi)(x-yi)$.
19. $(a+bi)^2-(a-bi)^2$.
20. $(1+\sqrt{3}i)(1-\sqrt{3}i)$.
21. $\dfrac{-1+\sqrt{-3}}{2} \cdot \dfrac{-1-\sqrt{-3}}{2}$.
22. $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^2$.
23. $\left(\dfrac{-1-\sqrt{-3}}{2}\right)^2$.
24. $(2a-i)(2a+i)$.
25. $(x^2+xi+1)(x^2-xi+1)$.
26. $(-x^3i-x^2+xi+1)(xi-1)$.
27. $(a^3-ab^2-a^2bi-b^3i)(a+bi)$.
28. $(a^3-ab^2+a^2bi-b^3i)(a-bi)$.
29. $(x^6-x^2y^4+x^4y^2i-y^6i)(x^2-y^2i)$.
30. $(x^6-x^2y^4-x^4y^2i+y^6i)(x^2+y^2i)$.
31. $(a^5-a^4bi+a^3b^2-a^2b^3i+ab^4-b^5i)(a+bi)$.
32. $(a^5+a^4bi+a^3b^2+a^2b^3i+ab^4+b^5i)(a-bi)$.
33. $(x^{10}+x^8y^2i+x^6y^4+x^4y^6i+x^2y^8+y^{10}i)(x^2-y^2i)$.
34. $(x^{10}-x^8y^2i-x^6y^4+x^4y^6i+x^2y^8-y^{10}i)(x^2+y^2i)$.
35. $(x^4-x^2y^2i+y^4)(x^4+x^2y^2i+y^4)$.

WRITTEN EXERCISES

Simplify the following indicated quotients:

1. $\dfrac{\sqrt{-3}}{\sqrt{-1}}$.
2. $\dfrac{\sqrt{-9}}{\sqrt{-4}}$.
3. $\dfrac{\sqrt{-16}}{\sqrt{4}}$.
4. $\dfrac{\sqrt{-4}+\sqrt{-9}}{\sqrt{-1}}$.
5. $\dfrac{\sqrt{-20}}{\sqrt{-4}}$.
6. $\dfrac{\sqrt{-8}+\sqrt{-32}}{\sqrt{-2}}$.
7. $\dfrac{1+i}{1-i}$.
8. $\dfrac{1-i}{1+i}$.
9. $\dfrac{2+3i}{1+i}$.
10. $\dfrac{3-2i}{3+2i}$.
11. $\dfrac{3+2i}{2-3i}$.
12. $\dfrac{1-4i}{4-i}$.
13. $\dfrac{1-3i}{3-i}$.
14. $\dfrac{i-2}{3+i}$.
15. $\dfrac{2}{1-\sqrt{-1}}$.
16. $\dfrac{3}{\sqrt{3}+\sqrt{-1}}$.
17. $\dfrac{1-\sqrt{-1}}{1+\sqrt{-1}}$.
18. $\dfrac{\sqrt{2}+\sqrt{-3}}{\sqrt{2}-\sqrt{-3}}$.
19. $\dfrac{5}{2-3\sqrt{-5}}$.
20. $\dfrac{x+y\sqrt{-1}}{x-y\sqrt{-1}}$.

HISTORICAL NOTE

The Imaginary Number had its origin entirely in the attempt to solve quadratic and higher equations. It was not until the time of **Argand** (1768–1825) and **Gauss** (1777–1855) that the imaginary was fully admitted to be a number. The notation i for $\sqrt{-1}$ was first used by Gauss.

Cardan (1545), who understood the solution of the cubic, and who showed that imaginary roots enter in pairs, that is, if a cubic has one imaginary root it has two, decided "not to commit himself to any explanation as to the meaning of these 'sophistic' quantities which he said were ingenious though useless."

During the eighteenth century **Euler** and others used the imaginary fully in mathematical operations without attaining a full insight into its nature.

In modern mathematics the imaginary plays many important rôles which are far beyond the scope of such a book as this. Even in electrical engineering, formulas containing the imaginary are in constant use.

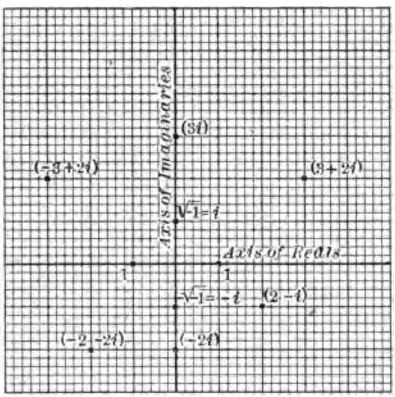
GRAPHIC REPRESENTATION OF COMPLEX NUMBERS

219. Graphs of Real Numbers. We have learned to represent real numbers by *points on a straight line*, — positive numbers on one side of the zero point, and negative numbers on the other side.

We have also learned to represent pairs of real numbers by *points in the plane*. See § 131.

We will now study a device for representing complex numbers in a similar manner.

220. The Axis of Reals and the Axis of Imaginaries. As in ordinary graphing, we construct two axes at right angles to each other. The horizontal line is called the **axis of reals**, and the vertical line, the **axis of imaginaries.**



Any purely real number, such as 4, or − 10, is laid off on the axis of reals. A pure imaginary is laid off on the axis of imaginaries.

Thus, the numbers i and $3i$ correspond to points on the imaginary axis one unit and three units, respectively, above the origin, while $-i$ and $-2i$ correspond to points on the same axis one unit and two units, respectively, below the origin.

A reason for making the vertical axis the axis of pure imaginaries appears in the following:

If the positive half of the axis of reals is rotated about the origin counter-clockwise through an angle of 180°, it is changed into the position of the negative half of this axis, and the point representing any positive number is changed into the position of the corresponding negative number.

But the positive numbers are also changed into corresponding negative numbers by multiplying each by -1. Hence multiplying by -1 may be interpreted graphically as *rotating the positive number line into the negative number line.*

Now by the definition of the imaginary unit, we have

$$(\sqrt{-1})^2 = \sqrt{-1} \times \sqrt{-1} = -1.$$

That is, multiplying by $\sqrt{-1}$ *twice in succession* is equivalent to multiplying by -1.

Hence, if $\sqrt{-1} \times \sqrt{-1} \times a$ changes any positive number a into $-a$, that is, rotates the positive real axis through 180°, we may interpret $\sqrt{-1} \times a$ as *rotating the positive real axis one half as far, that is, through* 90°, *or into the vertical position.*

Thus, $a\sqrt{-1}$ is represented graphically by a point on the vertical axis, at a distance a above the origin, and $-a\sqrt{-1}$ is represented by a corresponding point at a distance a below the origin.

221. Representation of Complex Numbers. Since a complex number, like $3+2i$, has a real part 3, and a pure imaginary part $2i$, we measure 3 units to the right along the real axis, and 2 units up parallel to the imaginary axis, and take the point (3, 2) thus located as the graphic representation of the complex number, $3+2i$.

Similarly, to represent $-2-2i$, we measure 2 units to the left along the real axis, and 2 units down parallel to the imaginary axis, and take the point $(-2, -2)$ thus located as the graphic representation of the complex number, $-2-2i$.

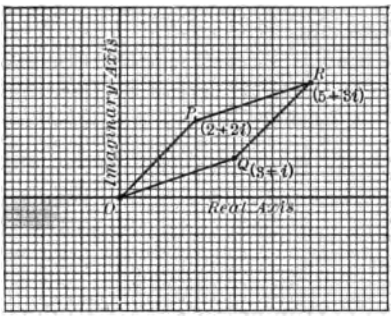
In this manner any complex number $a+bi$ is represented by the point whose abscissa is a, and whose ordinate is b.

Thus, *every point in the plane is the graphic representation of a certain complex number, and every complex number is represented by a certain point in the plane.*

222. The Complex Plane. This plane, in which the vertical axis is the axis of pure imaginaries, is called *the complex plane,* to distinguish it from the real plane in which real numbers are plotted on the vertical axis as well as on the horizontal axis.

223. Graphic Operations. By means of the graphic representation of complex numbers, it is possible to perform *graphically* all the operations of algebra on complex numbers. We will study graphic *addition* of such numbers. Graphic subtraction, multiplication, and division, and the finding of roots, are more complicated and are reserved for a more advanced course.

224. Geometric Representation of Addition. The sum of the numbers, $3+i$ and $2+2i$ is $5+3i$, and is represented by the point R shown in the figure.



The point R is obtained by connecting P and Q with the origin, and then drawing PR parallel to OQ and QR parallel to OP. This makes $OPRQ$ a parallelogram.

The construction for locating the point representing the *sum of any two complex numbers* is like the one just described. If P and Q are points representing the numbers $a+bi$, and $c+di$, then in the parallelogram $OPRQ$, the point R represents $(a+c)+(b+d)i$, provided O is the origin.

EXERCISES

1. On squared paper draw axes, and locate the points representing the following numbers: $2i$, $-4i$, 4, -3, $3+5i$, $-3+2i$, $-3-4i$, $2-3i$.

2. Construct the parallelogram representing the addition of $3+2i$ and $1+4i$.

3. Construct parallelograms representing the following indicated additions: $(2-2i)+(3+i)$; $(-1+2i)+(-2-3i)$.

CHAPTER XIV

QUADRATIC EQUATIONS

225. Quadratic Equations. Equations containing a square of the unknown quantity, and no higher power, are called *quadratic equations,* or equations of the second degree.

Thus, $x^2 = 4$, $x^2 + 7x = 0$ and $x^2 + 5x + 6 = 0$ are quadratic equations.

226. Solution by Factoring. A quadratic equation may frequently be solved by *factoring,* as in the following examples.

Example 1. Solve $x^2 - 5x + 6 = 0$.

Factoring, $(x - 3)(x - 2) = 0$.

Since a product is zero when one of its factors is zero, it follows that the roots of this equation are obtained by putting $x - 3 = 0$ and $x - 2 = 0$, and solving, thus obtaining $x = 2$ and $x = 3$.

Substituting these values in $(x-3)(x-2)=0$, we have $(3-3)(3-2) = 0 \cdot 1 = 0$, and $(2-3)(2-2) = -1 \cdot 0 = 0$.

Example 2. Solve $6x^2 + 5x = 6$.

Transposing and factoring, $(2x + 3)(3x - 2) = 0$.

From $2x + 3 = 0$, we have $x = -\frac{3}{2}$ and from $3x - 2 = 0$ we have $x = \frac{2}{3}$.

Example 3. Solve $x^2 + 7x = 0$.

Factoring, $x(x + 7) = 0$.

From $x + 7 = 0$ we have $x = -7$, and $x = 0$ gives the other root directly.

227. From these examples we get the following **rule**:

To solve a quadratic equation by factoring,

(1) *Transpose all terms to the left member.*

(2) *Factor the left member.*

(3) *Put each factor equal to zero, and solve for x.*

ORAL EXERCISES

Solve the following equations by factoring:

1. $x^2 - 2x + 1 = 0.$
2. $x^2 - 3x + 2 = 0.$
3. $x^2 - 4x + 3 = 0.$
4. $x^2 - 5x + 4 = 0.$
5. $x^2 + x - 2 = 0.$
6. $x^2 - x - 2 = 0.$
7. $x^2 - 2x - 3 = 0.$
8. $x^2 + 2x - 3 = 0.$
9. $x^2 - 3x - 4 = 0.$
10. $x^2 + 3x - 4 = 0.$
11. $x^2 + 3x + 2 = 0.$
12. $x^2 + 5x + 4 = 0.$
13. $x^2 - 5x + 4 = 0.$
14. $x^2 + 4x - 5 = 0.$
15. $x^2 - 4x - 5 = 0.$
16. $x^2 + 6x + 5 = 0.$

228. Many equations of higher degree than the second may be solved by factoring.

Example 1. Solve $2x^3 - x^2 - 5x - 2 = 0.$

We find by the *factor theorem* that $x - 2$ and $x + 1$ are factors of the left member. By division, the remaining factor is found to be $2x + 1$.

Hence, we have $(x - 2)(x + 1)(2x + 1) = 0.$

Hence, $x = 2$, $x = -1$, and $x = -\frac{1}{2}$ are roots of the equation.

Example 2. Solve $x^3 - 9x^2 + 26x - 24 = 0.$

By the *factor theorem* we find that $x - 4$ is a factor. Dividing, we find that $x^2 - 5x + 6$ is the quotient.

Hence the equation reduces to $(x - 4)(x - 2)(x - 3) = 0$, and the roots are $x = 4$, $x = 2$, $x = 3$.

WRITTEN EXERCISES

Solve the following equations by factoring:

1. $x^2 + 5x = -6.$
2. $x^2 - 4x = 21.$
3. $x^2 - 5x = 24.$
4. $x^2 + 9x = 10.$
5. $x^2 - 9x = 10.$
6. $x^2 + 4x = 60.$
7. $x^2 - 5x = -4.$
8. $x^2 - 5x = 36.$
9. $x^2 + 7x = 60.$
10. $x^2 - 10x = 11.$

Solve the following equations by factoring:

11. $2x^2 - x - 3 = 0.$
12. $3x^2 - 13x - 10 = 0.$
13. $3x^2 + 11x = 4.$
14. $2x^2 + 11x = 6.$
15. $6x^2 + x = 2.$
16. $5x^2 + 13x = 6.$
17. $4x^2 + 4x = 3.$
18. $8x^2 - 2x = 15.$
19. $7x^2 + 11x = 6.$
20. $4x^2 - 10x = 24.$
21. $x^3 - x^2 - 17x - 15 = 0.$
22. $x^3 = -2x^2 + 5x + 6.$
23. $x^3 - 12x^2 + 35x - 24 = 0.$
24. $x^3 - 4x^2 - 39x - 54 = 0.$
25. $x^3 + 6x^2 - 31x - 36 = 0.$
26. $x^3 + 3x^2 = 28x.$
27. $5x^3 + 9x^2 = 2x.$
28. $10x^3 - 5x^2 = 15x.$
29. $x^3 + 8x^2 - 3x = 90.$
30. $x^3 - 11x^2 + 34x = 24.$

229. Completing the Square. Any quadratic equation may be solved by *completing the square.*

As a preliminary step consider the trinomial square.

From $(x + a)^2 = x^2 + 2ax + a^2$, we see that the third term is the square of half the coefficient of x in the second term.

Hence, if we have given only the two terms $x^2 + 2ax$, we must add a^2 in order to *complete the square.*

E.g. to complete the square in $x^2 + 6x$ we add the square of half the coefficient of x, or $3^2 = 9$. That is, $x^2 + 6x + 9$ is *a complete square.*

In general, to complete the square in $x^2 + px$, we add $\left(\frac{1}{2}p\right)^2 = \frac{p^2}{4}$, since $x^2 + px + \frac{p^2}{4}$ is a complete square.

ORAL EXERCISES

Complete the trinomial square in each of the following:

1. $x^2 + 2x.$
2. $x^2 + 4x.$
3. $x^2 + 6x.$
4. $x^2 + 8x.$
5. $x^2 + 3x.$
6. $x^2 + 5x.$
7. $(3x)^2 + 2(3x).$
8. $(2x)^2 + 4(2x).$
9. $16x^2 + 2(4x).$

10. Complete the square in each of the above exercises, after replacing the sign + by −.

230. Solution of a Quadratic by Completing the Square.

Example 1. Solve the equation: $x^2 + 6x + 5 = 0$. (1)

Transposing in (1), $x^2 + 6x = -5$. (2)

To complete the square add $3^2 = 9$ to both members, obtaining

$$x^2 + 6x + 3^2 = 3^2 - 5 = 4. \quad (3)$$

Taking square roots of both sides, $x + 3 = \pm\sqrt{4} = \pm 2$.

Hence, $x = -3 + 2 = -1$,

and $x = -3 - 2 = -5$.

Example 2. Solve the equation:

$$x^2 - 12x + 42 = 56. \quad (1)$$

Transposing, $x^2 - 12x = 14$. (2)

Completing the square by adding $(\frac{12}{2})^2 = 6^2 = 36$ to both sides,

$$x^2 - 12x + 36 = 14 + 36 = 50. \quad (3)$$

Taking square roots, $x - 6 = \pm\sqrt{50} = \pm 5\sqrt{2}$. (4)

Transposing, $x = 6 \pm 7.071$. (5)

Hence, $x = 6 + 7.071 = 13.071$,

and also, $x = 6 - 7.071 = -1.071$.

The steps involved in the above solutions are:

(1) *Write the equation in the form* $x^2 + px = q$.

(2) *Complete the square by adding* $(\frac{1}{2}p)^2$ *to each member.*

(3) *Take the square root of both members of this equation.*

(4) *Solve each of the first degree equations thus obtained.*

WRITTEN EXERCISES

In solving the following quadratic equations the result may in each case be reduced so that the number remaining under the radical sign shall be 2, 3, or 5. Use $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$, and get results to two decimal places.

1. $x^2 - 4x = 8$.	6. $x^2 - 12x = 12$.	11. $8 = x^2 + 4x$.
2. $x^2 = 3 - 6x$.	7. $x^2 - 8x = -14$.	12. $23 - 6x = x^2$.
3. $4x = 16 - x^2$.	8. $x^2 = 2x + 1$.	13. $7 + 2x = x^2$.
4. $x^2 + 6x = 9$.	9. $x^2 - 4x = 16$.	14. $25 - x^2 = 5x$.
5. $x^2 + 6x = 11$.	10. $x^2 = 23 + 4x$.	15. $x^2 + 3x = 9$.

231. The Hindu Method of Completing the Square. In case the coefficient of x^2 is not unity, as in $3x^2+8x=4$, both members may be divided by this coefficient, and the solution is then like that of Examples 1 and 2 on page 170.

However, the following method is sometimes desirable:

$$3x^2+8x=4. \tag{1}$$

Multiplying each member of equation (1) by $4 \cdot 3 = 12$, we get

$$36x^2+96x=48. \tag{2}$$

This can now be written in the form $x^2+px=q$, namely, $(6x)^2+16(6x)=48$, in which $p=16$ and $6x$ is the unknown. Hence, we add $(\frac{16}{2})^2=8^2=64$ to complete the square, and get

$$(6x)^2+16(6x)+64=48+64=112. \tag{3}$$

Taking square roots, $\quad 6x+8=\pm\sqrt{112}=\pm 4\sqrt{7}. \qquad (4)$

Hence, $\quad 6x=-8\pm 4\sqrt{7},$

and $\quad x=-\frac{4}{3}\pm\frac{2}{3}\sqrt{7}. \qquad (5)$

In this solution both sides were multiplied by 4 times the original coefficient of x^2, and then the number added to complete the square was found to be *the square of the original coefficient of x.*

The advantage of this form of solution is that fractions are avoided until the last step, and *the number added to complete the square is equal to the square of the coefficient of x in the original equation.*

This is called the **Hindu method** of completing the square because it was first used by the Hindus.

NOTE. — Fractions would also be avoided in the above solution if equation (1) were multiplied by 3 instead of $4 \cdot 3$. This is the case only when the coefficient of x is an *even* number.

Any quadratic equation may be solved by the following

Rule. 1. *Write the equation in the form* $ax^2+bx=-c$.

2. *Multiply both sides by* $4a$, *and put the equation thus:*

$$(2ax)^2+2b(2ax)=-4ac.$$

3. *Complete the square by adding* b^2 *to both sides.*

4. *Solve as though* $2ax$ *were the unknown.*

EXERCISES

In the solution of the following equations, the roots which contain surds may be left in the simplified radical form.

1. $2x^2 + 3x = 2$.
2. $3x^2 + 5x = 2$.
3. $3x = 9 - 2x^2$.
4. $6x + 1 = -3x^2$.
5. $2x^2 = 5x + 3$.
6. $4x = 2x^2 - 1$.
7. $2x^2 - 3x = 14$.
8. $3x^2 = 9 + 2x$.
9. $4x^2 = 2x + 1$.
10. $6x - 1 = 3x^2$.
11. $2x^2 + 4x = 23$.
12. $3x^2 - 7 = 4x$.
13. $2x^2 - 5 = 3x$.
14. $4x^2 = 6x - 1$.
15. $2x = 1 - 5x^2$.
16. $3x - 20 = -2x^2$.
17. $2x + 3x^2 = 9$.
18. $4x^2 - 1 = 3x$.
19. $4x = 7 - 2x^2$.
20. $2x + 1 = 5x^2$.
21. $3x^2 + 4x = 7$.
22. $3x + 9 = 2x^2$.
23. $2x - 1 = -4x^2$.
24. $5x^2 + 16x = -2$.
25. $4x^2 + 1 = 8x$.
26. $2x^2 - 3x = 20$.
27. $2x^2 - 3 = -5x$.
28. $3x^2 + 4x = 8$.
29. $10 - 4x = 5x^2$.
30. $1 + 4x^2 = -6x$.
31. $5 - 3x = 2x^2$.
32. $7 + 4x = 2x^2$.
33. $6x^2 + 12x = 2$.
34. $6x^2 - 12x = -2$.
35. $6x^2 + 12x = -2$.
36. $6x^2 - 12x = 2$.
37. $3x^2 + 2x = 5$.
38. $2 + 3x = 2x^2$.
39. $8x + 1 = -4x^2$.
40. $8 + 4x = 3x^2$.
41. $10 + 4x = 5x^2$.
42. $2 + 5x = 3x^2$.
43. $3x + 14 = 2x^2$.
44. $3x^2 - 2x = 5$.
45. $2x^2 + 4x = 1$.
46. $3x - 1 = -4x^2$.
47. $23 + 4x = 2x^2$.
48. $3x - 1 = 2x^2$.
49. $5x^2 - 8x = 3$.
50. $7x^2 + 5x - 9 = 0$.
51. $6x^2 - 11x + 3 = 0$.
52. $3x^2 - 14x + 6 = 0$.

232. Solution of Quadratics by Formula. A *formula* for solving *any* quadratic may be obtained as follows:

Example. Solve $ax^2 + bx + c = 0$. (1)

Solution.

Transposing, $ax^2 + bx = -c.$

Multiplying by $4a$, $4a^2x^2 + 4abx = -4ac.$ (2)

Equation (2) may be written in the form

$$(2ax)^2 + 2b(2ax) = -4ac.$$

Completing the square as if $2ax$ were the unknown,

$$(2ax)^2 + 2b(2ax) + b^2 = b^2 - 4ac. \quad (3)$$

Taking square roots, $2ax + b = \pm\sqrt{b^2 - 4ac}.$ (4)

Transposing, $2ax = -b \pm \sqrt{b^2 - 4ac}.$

Dividing by $2a$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$ (5)

Calling the two values of x in the result x_1 and x_2, we have:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \quad (6)$$

Any quadratic equation may be reduced to the form of (1) and may therefore *be solved by substituting in the formulas* 6.

233. Method of Checking Results. A convenient method of checking the roots of a quadratic is obtained by finding the relations between the roots and the coefficients of the equation,

$$ax^2 + bx + c = 0.$$

By addition we obtain, $x_1 + x_2 = \dfrac{-2b}{2a} = \dfrac{-b}{a}.$ See (6), § 232.

By multiplication, $x_1x_2 = \dfrac{(-b + \sqrt{b^2 - 4ac})}{2a} \dfrac{(-b - \sqrt{b^2 - 4ac})}{2a}$

$$= \frac{(b - \sqrt{b^2 - 4ac})(b + \sqrt{b^2 - 4ac})}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}.$$

Hence in the equation $ax^2 + bx + c = 0$, *the sum of the roots equals* $-\dfrac{b}{a}$, *and the product of the roots equals* $\dfrac{c}{a}$.

Example. Solve $2x^2 - 4x + 1 = 0.$

Substituting $a = 2$, $b = -4$, $c = 1$, in the formula, we get

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2},$$

from which $x_1 = \frac{2 + \sqrt{2}}{2}$ and $x_2 = \frac{2 - \sqrt{2}}{2}.$

Check. $x_1 + x_2 = \frac{2 + \sqrt{2}}{2} + \frac{2 - \sqrt{2}}{2} = \frac{4}{2} = 2 = -\left(\frac{-4}{2}\right) = -\frac{b}{a}$

and $x_1 x_2 = \frac{2 + \sqrt{2}}{2} \cdot \frac{2 - \sqrt{2}}{2} = \frac{4 - 2}{4} = \frac{1}{2} = \frac{c}{a}.$

WRITTEN EXERCISES

By means of the formula (§ 232), find the solutions of each of the following equations and check each by the method of § 233:

1. $2x^2 - 3x - 4 = 0.$
2. $3x^2 + 2x - 1 = 0.$
3. $3x^2 - 2x - 1 = 0.$
4. $4x^2 + 6x + 1 = 0.$
5. $x^2 - 7x + 12 = 0.$
6. $5x^2 + 8x + 3 = 0.$
7. $5x^2 - 8x + 3 = 0.$
8. $5x^2 + 8x - 3 = 0.$
9. $5x^2 - 8x - 3 = 0.$
10. $2x - 3x^2 + 7 = 0.$
11. $3x - 9x^2 + 1 = 0.$
12. $7x^2 - 3x - 2 = 0.$
13. $6x^2 + 7x + 1 = 0.$
14. $4x^2 + 5x - 3 = 0.$
15. $4x^2 - 5x - 3 = 0.$
16. $8x^2 + 3x - 5 = 0.$
17. $7x^2 + x - 3 = 0.$
18. $7x^2 - x - 4 = 0.$
19. $x^2 - 2ax = 3b - a^2.$
20. $x^2 - 6ax = 49c^2 - 9a^2.$
21. $x^2 - \frac{mx}{2} + 2mn = 4nx.$
22. $x^2 - 2ax + 4ab = b^2 + 3a^2.$
23. $x^2 - abx + a^2b - ax = ab^2 - bx.$
24. $x^2 + 9 - c = 6x.$
25. $nx^2 + m^2n = mn^2x + mx.$
26. $x^2 + 2a^2 + 3a - 2 = (3a + 1)x.$

FRACTIONAL EQUATIONS LEADING TO QUADRATICS

234. When a quadratic is obtained by clearing an equation of fractions it may happen that one of the roots of this quadratic will fail to satisfy the original equation. See § 102. *This will be the case if any denominator in the given equation becomes zero on substituting a root of the quadratic.* Otherwise, both roots will satisfy the given equation.

Example. Solve $$\frac{x}{x-1}+\frac{x-1}{x}=\frac{x^2+x-1}{x(x-1)}.$$

Solution. Clearing of fractions by multiplying both members by $x(x-1)$,

$$x^2+x^2-2x+1=x^2+x-1.$$

Transposing, $$x^2-3x+2=0.$$

Factoring, $$(x-1)(x-2)=0.$$

Hence, $$x=1, \quad \text{and } x=2.$$

But $x=1$ reduces two denominators in the original equation to zero. Hence $x=1$ is not a root of the given equation.

$x=2$ is a root, as may be shown by substituting.

Thus, $$\frac{2}{2-1}+\frac{2-1}{2}=2+\tfrac{1}{2}=\tfrac{5}{2}.$$

WRITTEN EXERCISES

Solve the following equations, making sure that no denominator is reduced to zero by a supposed solution:

1. $\dfrac{3x^3+3}{3x^2-7x+3}=x-7.$

2. $\dfrac{x+2}{x-2}=2x+3.$

3. $\dfrac{3}{2x^2-x-1}+\dfrac{5}{x^2-1}+\dfrac{1}{x+1}=0.$

4. $\dfrac{2x}{2x-1}+\dfrac{x}{x+1}-\dfrac{3x}{x-1}=-1.$

5. $\dfrac{1}{3(x-1)}-\dfrac{1}{x^2-1}=\dfrac{1}{4}.$

6. $\dfrac{2a-1}{a}+\dfrac{1}{2}=\dfrac{3a}{3a-1}.$

7. $\dfrac{2}{x-1}+\dfrac{3}{x-2}=\dfrac{6}{x-3}.$

8. $\dfrac{1}{a-x}-\dfrac{1}{a+x}=-\dfrac{3+x^2}{a^2-x^2}.$

9. $\frac{2}{x-10}+10=x+\frac{2}{10-x}$.

10. $\frac{6-x}{x-4}+\frac{x-4}{6-x}=\frac{5}{2}$.

11. $\frac{a}{2a-1}+\frac{24}{4a^2-1}=\frac{2(a-4)}{2a+1}-\frac{1}{9}$.

12. $\frac{a}{a-1}+\frac{a-1}{a}=\frac{a^2+a-1}{a^2-a}$.

13. $\frac{1}{a^2-4}-\frac{3}{2-a}=1+\frac{1}{3(a+2)}$.

14. $\frac{(a-x)(x-b)}{(a-x)-(x-b)}=x$.

15. $\frac{x+m-2n}{x+m+2n}=\frac{n+2m-2x}{n-2m+2x}$.

16. $\frac{1}{x-2}+\frac{7x}{24(x+2)}=\frac{5}{x^2-4}$.

17. $\frac{x+a}{x-a}+\frac{x-a}{x+a}=\frac{2(a^2+1)}{(1+a)(1-a)}$.

18. $\frac{x-m}{x+m}=\frac{n-x}{n+x}$.

19. $\frac{1}{x-a}-\frac{2a}{x^2-a^2}=b$.

20. $\frac{4}{3x+1}+\frac{4(3x-1)}{2x+1}=\frac{2x+1}{3x+1}$.

21. $\frac{2x+3}{2(2x-1)}+\frac{7-3x}{3x-4}+\frac{x-7}{2(x+1)}=0$.

22. $\frac{y^2+2y-2}{y^2+5y+6}+\frac{y}{y+3}=\frac{y}{y+2}$.

23. $\frac{5}{2x+3}+\frac{7}{3x-4}=\frac{8x^2-13x-64}{6x^2+x-12}$.

24. $\frac{27}{x^3+27}-\frac{3}{x^2-3x+9}+\frac{1}{x+3}=0$.

25. $\frac{1}{x^3-1}-\frac{2}{x^2+x+1}=\frac{1}{x-1}$.

26. $\frac{a^2c}{x^3-a^3}-\frac{b}{x^2+xa+a^2}=\frac{c}{x-a}$.

IRRATIONAL EQUATIONS LEADING TO QUADRATICS

235. Many equations containing radicals may be solved by means of *quadratics.* Sometimes they even reduce to *linear equations,* as in the first example below.

Example 1. Solve $\dfrac{\sqrt{4x+1}-\sqrt{3x-2}}{\sqrt{4x+1}+\sqrt{3x-2}}=\dfrac{1}{5}.$

Solution. Clearing of fractions,

$$5\sqrt{4x+1}-5\sqrt{3x-2}=\sqrt{4x+1}+\sqrt{3x-2}.$$

Transposing, and combining similar radicals,

$$2\sqrt{4x+1}=3\sqrt{3x-2}.$$

Squaring both members, $4(4x+1)=9(3x-2)$.

Transposing and solving, we find $x=2$.

Check. Substituting $x=2$ in the given equation,

$$\frac{\sqrt{9}-\sqrt{4}}{\sqrt{9}+\sqrt{4}}=\frac{3-2}{3+2}=\frac{1}{5}.$$

Example 2. Solve $\dfrac{6-x}{\sqrt{6-x}}-\sqrt{3}=\dfrac{x-3}{\sqrt{x-3}}.$

Solution. We see that $\quad 6-x=\sqrt{6-x}\times\sqrt{6-x},$

and $\quad x-3=\sqrt{x-3}\times\sqrt{x-3}.$

Hence each fraction can be reduced to lower terms, thus,

$$\frac{\sqrt{6-x}\times\cancel{\sqrt{6-x}}}{\cancel{\sqrt{6-x}}}-\sqrt{3}=\frac{\sqrt{x-3}\times\cancel{\sqrt{x-3}}}{\cancel{\sqrt{x-3}}},$$

or
$$\sqrt{6-x}-\sqrt{3}=\sqrt{x-3}.$$

Squaring both sides, $6-x-2\sqrt{3}\sqrt{6-x}+3=x-3$.

Transposing and dividing by 2, $\quad x-6=-\sqrt{3}\,\sqrt{6-x}.$

Squaring again, $\quad x^2-12x+36=3(6-x).$

Transposing, $\quad x^2-9x+18=0.$

Factoring, $\quad (x-6)(x-3)=0.$

Hence, $\quad x=6$, and $x=3$.

But neither of these values will satisfy the given equation, since $x=6$ makes one denominator zero, and $x=3$ makes the other zero. Hence this equation *has no solution.*

Example 3. Solve $\sqrt{x-1}+\sqrt{4-x}=\sqrt{3}$.

Solution. It will be simpler to transpose first, so as to have one radical containing the unknown alone on one side, thus

$$\sqrt{x-1}-\sqrt{3}=-\sqrt{4-x}.$$

Squaring both members, $x-1-2\sqrt{3}\sqrt{x-1}+3=4-x$.

Transposing, $2x-2=2\sqrt{3}\sqrt{x-1}$,

or $x-1=\sqrt{3}\sqrt{x-1}$.

Squaring again and transposing, $x^2-5x+4=0$.

Solving, $x=4$, and $x=1$.

Both of the values satisfy the given equations.

236. The above examples suggest the following **rule** for solving irrational equations:

(1) *If a fraction of the form* $\dfrac{a-b}{\sqrt{a}-\sqrt{b}}$ *is involved, this should be reduced by dividing numerator and denominator by* $\sqrt{a}-\sqrt{b}$ *before clearing of fractions.*

(2) *If a fraction of the form* $\dfrac{a+b}{\sqrt{a+b}}$ *is involved, this should be reduced by dividing both numerator and denominator by* $\sqrt{a+b}$, *obtaining* $\sqrt{a+b}$.

(3) *After clearing of fractions, transpose terms so as to leave one radical alone in one member.*

(4) *Square both members, and if the resulting equation still contains radicals, transpose and square as before.*

(5) *In every case verify all results by substituting in the given equation.*

The equation should be looked over with care at the outset to see whether it may be simplified by using (1) or (2) of the above rule.

Thus in Example 3 on the next page, $\dfrac{by-1}{\sqrt{by}+1}$ may be thus simplified, giving $\sqrt{by}-1$.

Similarly, in Example 11, $\dfrac{x+5}{\sqrt{x+5}}=\sqrt{x+5}$.

Likewise, note Examples 7, 8, 12, and 13.

EXERCISES

Solve the following equations and check all solutions:

1. $\sqrt{x^2+7x-2}-\sqrt{x^2-3x+6}=2.$

2. $\sqrt{3y}-\sqrt{3y-7}=\dfrac{5}{\sqrt{3y-7}}.$

3. $\dfrac{by-1}{\sqrt{by}+1}=\dfrac{\sqrt{by}-1}{2}+4.$

4. $\sqrt{5x-19}+\sqrt{3x+4}=9.$

5. $\dfrac{\sqrt{x^2+a^2}-x}{\sqrt{x^2+a^2}+x}=2.$

6. $\sqrt{a+\sqrt{ax+x^2}}=\sqrt{a}-\sqrt{x}.$

7. $\dfrac{y-l}{\sqrt{y}+\sqrt{l}}=\dfrac{\sqrt{y}-\sqrt{l}}{3}+2\sqrt{l}.$

8. $\sqrt{5x}-3=-\dfrac{5x-1}{\sqrt{5x}+1}.$

9. $\dfrac{4}{x}-\dfrac{\sqrt{4-x^2}}{x}=\sqrt{3}.$

10. $\dfrac{4+x+\sqrt{8x+x^2}}{4+x-\sqrt{8x+x^2}}=4.$

11. $\dfrac{5-x}{\sqrt{5-x}}+\dfrac{x+5}{\sqrt{x+5}}=2.$

12. $\dfrac{x-a}{\sqrt{x}-\sqrt{a}}=\dfrac{\sqrt{x}+\sqrt{a}}{2}+2\sqrt{a}.$

13. $\dfrac{m-y}{\sqrt{m-y}}-\sqrt{m-n}=-\dfrac{y-n}{\sqrt{y-n}}.$

14. $2\sqrt{x-a}+3\sqrt{2x}=\dfrac{7a+5x}{\sqrt{x-a}}.$

15. $\sqrt{x-3}+\sqrt{x+9}=\sqrt{x+18}+\sqrt{x-6}.$

16. $\sqrt{x+7}-\sqrt{x-1}=\sqrt{x+2}+\sqrt{x-2}.$

17. $a\sqrt{y+b}-c\sqrt{b-y}=\sqrt{b(a^2+c^2)}.$

18. $y\sqrt{y-c}-\sqrt{y^3+c^3}+c\sqrt{y+c}=0.$

19. $\dfrac{\sqrt{x}}{\sqrt{m}}+\dfrac{\sqrt{m}}{\sqrt{x}}=\dfrac{\sqrt{m}}{\sqrt{n}}+\dfrac{\sqrt{n}}{\sqrt{m}}.$

HIGHER EQUATIONS REDUCIBLE TO QUADRATICS

237. Equations in the Quadratic Form. If an equation contains a certain expression and also the square of this expression, then the equation is a *quadratic in the given expression.*

Thus, $x^4 + 7x^2 = 12$ is a quadratic in x^2.

$2x + 7 + 3\sqrt{2x+7} = 9$ is a quadratic in $\sqrt{2x+7}$.

Such equations may often be solved by quadratic methods, even when of higher degree than the second.

Example 1. Solve $x^4 + 7x^2 = 44$.

This may be written $(x^2)^2 + 7(x^2) = 44,$
which is a quadratic in x^2. Solving for x^2, we find

$$(x^2 - 4)(x^2 + 11) = 0.$$

Hence, $x^2 = 4$ and $x^2 = -11,$

and $x = \pm 2$, and $x = \pm\sqrt{-11}$.

Example 2. Solve $x + 2 + 3\sqrt{x+2} = 18$.

Since $x + 2$ is the square of $\sqrt{x+2}$, this is a quadratic in $\sqrt{x+2}$.
Putting $\sqrt{x+2} = z$ for simplicity in solving, we have

$$z^2 + 3z - 18 = 0.$$

Solving, $z = 3$, and $z = -6$.

Replacing the value of z, we have

$$\sqrt{x+2} = 3, \text{ and } \sqrt{x+2} = -6.$$

Squaring, $x + 2 = 9$, and $x + 2 = 36$.

Hence, $x = 7$, and $x = 34$.

$x = 7$ satisfies the given equation, but $x = 34$ does not.

Example 3. Solve $(2x^2 - 1)^2 - 5(2x^2 - 1) - 14 = 0$.

This is a quadratic in the expression $2x^2 - 1$.
Putting $2x^2 - 1 = z$, we have

$$z^2 - 5z - 14 = 0.$$

Solving for z, $z = -2$, and $z = 7$.

Replacing the values of z,

$$2x^2 - 1 = -2, \text{ and } 2x^2 - 1 = 7.$$

Solving for x, $x = \pm\frac{1}{2}\sqrt{-2}$ and $x = \pm 2$.

All four values of x satisfy the given equation.

Example 4. Solve

$$x^2 - 17x + 40 - 2\sqrt{x^2 - 17x + 69} = -26. \qquad (1)$$

Add 29 to each member, obtaining

$$x^2 - 17x + 69 - 2\sqrt{x^2 - 17x + 69} = 3. \qquad (2)$$

Equation (2) is a quadratic in $\sqrt{x^2 - 17x + 69}$.

Putting $\sqrt{x^2 - 17x + 69} = z$,

we have $z^2 - 2z - 3 = 0. \qquad (3)$

Solving for z, $z = 3$, and $z = -1$.

Replacing the values of z,

$$\sqrt{x^2 - 17x + 69} = 3, \text{ and } \sqrt{x^2 - 7x + 69} = -1.$$

Squaring, $x^2 - 17x + 69 = 9$, and $x^2 - 17x + 69 = 1$.

Transposing, $x^2 - 17x + 60 = 0, \qquad (4)$

and $x^2 - 17x + 68 = 0. \qquad (5)$

From (4), $(x - 5)(x - 12) = 0$ or $x = 5$, $x = 12$.

From (5), $x = \dfrac{17 \pm \sqrt{17}}{2}$, but these values of x do not satisfy (2).

WRITTEN EXERCISES

Solve the following equations:

1. $x^4 - 5x^2 + 6 = 0.$
2. $5x - 4 - 2\sqrt{5x - 4} = 63.$
3. $(2 - x + x^2)^2 + x^2 - x = 18.$
4. $a^2 - 3a + 4 - 3\sqrt{a^2 - 3a + 4} = -2.$
5. $3a^6 - 57a^3 - 648 = 0.$
6. $x^2 - 8x + 16 + 6\sqrt{x^2 - 8x + 16} = 40.$
7. $\left(a + \dfrac{2}{a}\right)^2 + 4\left(a + \dfrac{2}{a}\right) = 21.$
8. $a^8 - 97a^4 + 1296 = 0.$
9. $a^2 - 3a + 4 + \sqrt{a^2 - 3a + 15} = 19.$
10. $(5x - 7 + 3x^2)^2 + 3x^2 + 5x - 247 = 0.$
11. $\sqrt[3]{7x - 6} - 4\sqrt[6]{7x - 6} + 4 = 0.$
12. $x^6 + 19x^3 = 216.$

RELATIONS BETWEEN THE ROOTS AND COEFFICIENTS OF A QUADRATIC

238. The Sum and Product of the Roots. We have seen (§ 233) that in the solution of $ax^2 + bx + c = 0$ the two roots,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

are such that $x_1 + x_2 = -\frac{b}{a}$ and $x_1x_2 = \frac{c}{a}$.

E.g. in $2x^2 - 3x - 8 = 0$, $x_1 + x_2 - \frac{-3}{2} = \frac{3}{2}$ and $x_1x_2 = -\frac{8}{2} = -4$.

239. The Discriminant of the Quadratic Equation. The expression $b^2 - 4ac$ under the radical sign in the two values of x above is called the *discriminant of the quadratic.* Its value determines the nature of the roots of the equation as follows:

(1) If $b^2 > 4ac$, that is, if $b^2 - 4ac > 0$, then both x_1 and x_2 are *real* (see § 151), and they are distinct since

$$x_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac$ is a perfect square, then both x_1 and x_2 are *rational* (see § 15).

(2) If $b^2 = 4ac$, that is, if $b^2 - 4ac = 0$, then the values of x_1 and x_2 are the same, since $x_1 = -\frac{b}{2a} + 0$ and $x_2 = -\frac{b}{2a} - 0$.

(3) If $b^2 < 4ac$, that is, if $b^2 - 4ac < 0$, then both x_1 and x_2 are *complex numbers,* since $\sqrt{b^2 - 4ac}$ is *imaginary* (see § 212).

240. Summary. *If x_1 and x_2 are the roots of $ax^2 + bx + c = 0$, then we have*

$$x_1 + x_2 = -\frac{b}{a}, \text{ and } x_1x_2 = \frac{c}{a}, \qquad (1)$$

x_1 and x_2 are real and distinct, if $b^2 - 4ac > 0$, (2)

x_1 and x_2 are real and equal, if $b^2 - 4ac = 0$, (3)

x_1 and x_2 are imaginary, if $b^2 - 4ax < 0$. (4)

By means of the results (1) to (4), we may determine the character of the roots of a quadratic without solving it.

Example 1. Determine the character of the roots of

$$8x^2 - 3x - 9 = 0.$$

Since $b^2 - 4ac = 9 - 4 \cdot 8(-9) = 297 > 0$, the roots are *real* and *distinct*. Since $b^2 - 4ac$ is not a perfect square, the roots are *irrational*.

Since $\frac{c}{a} = -\frac{9}{8} = x_1x_2$, the roots have *opposite signs*.

Since $-\frac{b}{a} = \frac{3}{8} = x_1 + x_2$, the *positive root is greater* in absolute value.

Example 2. Examine $3x^2 + 5x + 2 = 0$.

Since $b^2 - 4ac = 25 - 4 \cdot 3 \cdot 2 = 1 > 0$, the roots are *real* and *distinct*. Since $b^2 - 4ac$ is a perfect square, the roots are *rational*.

Since $\frac{c}{a} = \frac{2}{3} = x_1x_2$, the roots have the *same sign*.

Since $-\frac{b}{a} = -\frac{5}{3} = x_1 + x_2$, the roots are *both negative*.

Example 3. Examine $x^2 - 14x + 49 = 0$.

Since $b^2 - 4ac = 196 - 4 \cdot 49 = 0$, the roots are *real* and *equal*.

Example 4. Examine $x^2 - 7x + 15 = 0$.

Since $b^2 - 4ac = 49 - 4 \cdot 15 = -11$, the roots are *imaginary*.

ORAL EXERCISES

1. Find the value of $b^2 - 4ac$ in $2x^2 - 3x + 2 = 0$.

2. Find the sum and also the product of the roots in

$$3x^2 - 4x - 5 = 0.$$

3. Show that the roots of $2x^2 - 5x - 3 = 0$ have *opposite* signs. Which has the *greater* absolute value?

4. Show that the roots of $2x^2 + 8x - 3 = 0$ are *real*.

5. Show that the roots of $2x^2 - 7x + 3 = 0$ are *rational*.

6. Show that the roots of $3x^2 + 11x - 2 = 0$ are *real* but irrational.

7. Show that the roots of $2x^2 - 3x + 2 = 0$ are *imaginary*.

WRITTEN EXERCISES

Without solving, determine the character of the roots in each of the following:

1. $5x^2 - 4x - 5 = 0.$
2. $6x^2 + 4x + 2 = 0.$
3. $x^2 - 4x + 8 = 0.$
4. $2 + 2x^2 = 4x.$
5. $6x + 8x^2 = 9.$
6. $1 - a^2 = 3a.$
7. $6a - 30 = 3a^2.$
8. $6a^2 + 6 = 13a.$
9. $16m^2 + 4 = 16m.$
10. $25a^2 - 10a = 8.$
11. $20 - 13b - 15b^2 = 0.$
12. $10y^2 + 39y + 14 = 0.$
13. $3a^2 + 5a + 22 = 0.$
14. $3a^2 - 22a + 21 = 0.$
15. $5b^2 + 6b = 27.$
16. $6a - 17 = 11a^2.$

FORMATION OF EQUATIONS WHOSE ROOTS ARE GIVEN

241. If the roots of a quadratic equation are given, the equation may be formed by recalling the *solution by factoring*.

Example 1. Form the equation whose roots are 7 and -4.

Solution. Since a quadratic in the form $(x-a)(x-b) = 0$ has the roots $x = a$ and $x = b$, we see that $(x-7)(x-(-4)) = 0$ is an equation whose roots are 7 and -4.

Hence, $(x-7)(x+4) = x^2 - 3x - 28 = 0$ is the required equation.

Example 2. Form the equation whose roots are 2, 3, and 5.

Solution. The required equation is

$$(x-2)(x-3)(x-5) = x^3 - 10x^2 + 31x - 30 = 0.$$

ORAL EXERCISES

Form the equations whose roots are:

1. 1, 2.
2. 1, 3.
3. -2, 3.
4. 1, 4.
5. 2, 4.
6. 2, 6.
7. -1, -2.
8. -1, -3.
9. 2, -3.
10. -1, -4.
11. -2, -4.
12. -2, -6.
13. 3, 4.
14. -3, -4.
15. 4, -5.

WRITTEN EXERCISES

Form the equations whose roots are:

1. $3, -7$.
2. b, c.
3. $a, -b, -c$.
4. $5, -4, -2$.
5. $-5, -6$.
6. $a, -b$.
7. $-b+k, -b-k$.
8. $2, 3, 4, 5$.
9. $1, \frac{1}{2}, \frac{1}{3}, 3$.
10. $\sqrt{5}, -\sqrt{5}$.
11. $\sqrt{-1}, -\sqrt{-1}$.
12. $8+\sqrt{3}, 8-\sqrt{3}$.
13. $3+2\sqrt{-1}, 3-2\sqrt{-1}$.
14. $5-\sqrt{-1}, 5+\sqrt{-1}$.
15. $a-\sqrt{3}, a+\sqrt{3}$.
16. $\dfrac{-b+\sqrt{b^2-4ac}}{2a}, \dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

242. Factoring by Solving a Quadratic. An expression of the second degree in a single letter may be *resolved into factors*, each of the first degree in that letter, by solving a quadratic equation.

Example 1. Factor $6x^2-17x+5$.

This trinomial may be written, $6(x^2-\frac{17}{6}x+\frac{5}{6})$.

Solving the equation, $x^2-\frac{17}{6}x+\frac{5}{6}=0$, we find $x_1=\frac{1}{3}$ and $x_2=\frac{5}{2}$. Hence, by the factor theorem, § 70, $x-\frac{1}{3}$ and $x-\frac{5}{2}$ are factors of $x^2-\frac{17}{6}x+\frac{5}{6}$. And finally,

$$6(x^2-\tfrac{17}{6}x+\tfrac{5}{6})=6(x-\tfrac{1}{3})(x-\tfrac{5}{2})=3(x-\tfrac{1}{3})\cdot 2(x-\tfrac{5}{2})$$
$$=(3x-1)(2x-5).$$

This process is usually not needed when the factors are *rational*. It is particularly applicable when the factors are *irrational* or *imaginary*.

Example 2. Factor $3x^2+8x-7=3(x^2+\frac{8}{3}x-\frac{7}{3})$.

Solving the equation $x^2+\frac{8}{3}x-\frac{7}{3}=0$, we find,

$$x_1=\frac{-4+\sqrt{37}}{3} \text{ and } x_2=\frac{-4-\sqrt{37}}{3}.$$

Hence, as above:

$$3x^2+8x-7=3\left[x-\frac{-4+\sqrt{37}}{3}\right]\left[x-\frac{-4-\sqrt{37}}{3}\right]$$
$$=3\left[x+\frac{4}{3}-\frac{\sqrt{37}}{3}\right]\left[x+\frac{4}{3}+\frac{\sqrt{37}}{3}\right].$$

WRITTEN EXERCISES

Using the method of § 242, factor the following:

1. $x^2 - 4x + 8$.
2. $x^2 + 6x - 9$.
3. $a^2 + 3a - 1$.
4. $2x^2 - 4x + 2$.
5. $5x^2 - 4x - 5$.
6. $6x^2 + 4x + 2$.
7. $3a^2 - 6a + 30$.
8. $6a^2 - 13a + 6$.
9. $16m^2 - 32m + 15$.
10. $25a^2 - 10a - 8$.
11. $15b^2 + 13b - 20$.
12. $10y^2 + 39y + 14$.
13. $3a^2 + 5a + 22$.
14. $3a^2 - 22a + 21$.
15. $5b^2 + 6b - 27$.
16. $11a^2 - 6a + 17$.

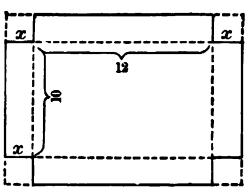
PROBLEMS IN ONE UNKNOWN LEADING TO QUADRATICS

243. Interpreting Results in Solving Problems. A problem which gives rise to a quadratic equation *may have two solutions*, in which case the two roots of the quadratic are the solutions.

If such a problem has *only one solution*, it is necessary to determine which of the two roots of the quadratic satisfies the conditions of the problem.

Example 1. The dimensions of a picture inside the frame are 10 by 12 inches. The area of the frame is 104 square inches. Find the width of the frame.

Solution. If x = width of frame, then $4x^2$ is the area of the corners, and $2 \cdot 12x + 2 \cdot 10x = 44x$ is the area of the rest of the frame.



Hence, $4x^2 + 44x = 104$,

or $x^2 + 11x - 26 = 0$.

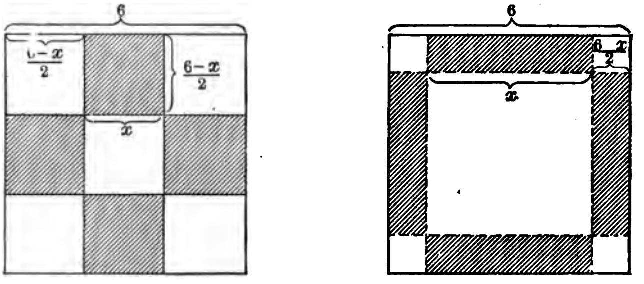
Hence, $x = 2$, or -13.

Since the width of a frame cannot be negative, the only solution which satisfies the problem is $x = 2$.

John Napier (1550–1617), Baron of Merchiston, Scotland, inventor of logarithms, was the first Englishman to publish important contributions to mathematics. Most of his life was spent on his family estates, where he took active part in political and religious controversies.

The work containing his treatment of logarithms was published in Latin in 1614, though he had confided the results of his discovery to the astronomer Tycho Brahe as early as 1594.

Example 2. In the accompanying figures, the large squares are 6 inches on each side. The shaded parts occupy $\frac{4}{9}$ of the area of each square. Find the width of the shaded strips.



Solution. Let $x =$ the length of a shaded strip in either figure. Then the total shaded area in each case is $4 \cdot x \cdot \frac{6-x}{2}$.

Hence, $$4x \cdot \frac{6-x}{2} = \frac{4}{9} \cdot 36 = 16,$$

or $$x^2 - 6x + 8 = 0,$$

or $$(x - 2)(x - 4) = 0.$$

Hence, $$x = 2, \text{ and } x = 4.$$

It is easily seen by studying the figures that this problem has two solutions, and that they are given by the two roots $x = 2$, and $x = 4$.

Example 3. Find three consecutive integers such that the sum of their squares is 110.

Solution. Let $x =$ the smallest integer.

Then $$x^2 + (x+1)^2 + (x+2)^2 = 110,$$

or $$3x^2 + 6x - 105 = 0.$$

$$x^2 + 2x - 35 = 0.$$

$$(x + 7)(x - 5) = 0.$$

Hence, $$x = 5, \text{ and } x = -7.$$

The positive integers 5, 6, 7, satisfy the conditions of the problem, as do also the negative integers -7, -6, -5.

If this problem were proposed in arithmetic, where the numbers used are all positive, the value $x = -7$ would not furnish a solution of the problem, whereas in algebra $x = -7$ is a legitimate solution.

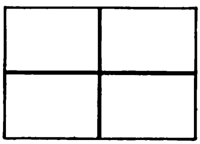
PROBLEMS IN ONE UNKNOWN LEADING TO QUADRATICS

Interpret both solutions of each quadratic:

1. A picture measured inside the frame is 18 by 24 inches. The area of the frame is 288 square inches. Find its width.

2. If in problem 1 the sides of the picture are a and b and the area of the frame c, find the width of the frame.

3. A rectangular park is 80 by 120 rods. Two driveways of equal width, one parallel to the longer and one to the shorter side, run through the park. What is the width of the driveways if their combined area is 591 square rods?



4. If in problem 3 the park is a rods wide and b rods long and the area of the driveways is c square rods, find their width.

5. A and B working together can do a piece of work in 6 days. A can do it alone in 5 days less than B. How long will it require each when working alone?

6. A and B working together can do a piece of work in n days. A does the work in d days less than B. How long will it require each when working alone?

7. The circumference of the rear wheel of a carriage is 4 feet greater than that of the front wheel. In running one mile the front wheel makes 110 revolutions more than the rear wheel. Find the circumference of each wheel.

8. Solve a general problem of which Example 7 is a special case, using b feet instead of one mile, letting the other numbers remain as they are in Example 7.

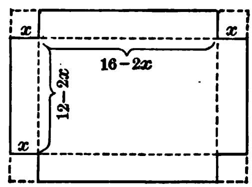
9. On her second westward trip the *Mauretania* traveled 625 knots in a certain time. If her speed had been 5 knots less per hour, it would have required $6\frac{1}{4}$ hours longer to cover the same distance. Find her speed per hour.

10. By increasing the speed a miles per hour, it requires b hours less to go c miles. Find the original speed. Show how problem 9 may be solved by means of the formula thus obtained.

11. A man can row 24 miles downstream and return in 9 hours. If his rate upstream is 4 miles per hour less than downstream, find the rate of the current and the rate of the boat in still water.

12. A man can row a miles downstream and return in b hours. If his rate upstream is c miles per hour less than downstream, find the rate of the current, and the rate of the boat in still water.

13. A rectangular sheet of tin, 12 by 16 inches, is made into an open box by cutting out a square from each corner and turning up the sides. Find the size of the square cut out if the volume of the box is 180 cubic inches.



The resulting equation is of the third degree. Solve it by factoring. See §§ 70, 228. Obtain three results and determine which are applicable to the problem.

14. A square piece of tin is made into an open box containing a cubic inches, by cutting from each corner a square whose side is b inches and then turning up the sides. Find the side of the original square. Also if $a = 18$, $b = 2$.

15. A rectangular piece of tin is 2 inches longer than it is wide. By cutting from each corner a square whose side is 1 inch and turning up the sides, an open box containing 18 cubic inches is formed. Find the dimensions of the original piece of tin.

16. The difference of the cubes of two consecutive integers is 397. Find the integers.

17. The upper base of a trapezoid is 8 and the lower base is 3 times the altitude. Find the altitude and the lower base if the area is 78.

The area of a trapezoid is equal to one half the sum of the bases multiplied by the altitude. See page 79.

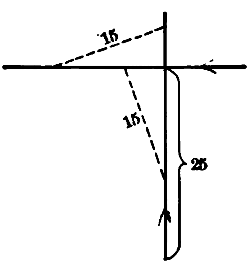
18. The lower base of a trapezoid is 4 greater than twice the altitude, and the upper base is $\frac{1}{2}$ the lower base. Find the two bases and the altitude if the area is $52\frac{1}{2}$.

19. The area of a regular hexagon inscribed in a circle is a. Find the radius of the circle.

20. One edge of a rectangular box is increased 6 inches, another 3 inches, and the third is decreased 4 inches, making a cube whose volume is 864 cubic inches greater than that of the original box. Find its dimensions.

21. Of two trains one runs 12 miles per hour faster than the other, and covers 144 miles in one hour less time. Find the speed of each train.

22. An automobile is running northward on a straight road at the rate of 25 miles per hour. A team is traveling westward on a straight road at the rate of 10 miles per hour. When the team crosses the north and south road the automobile is 25 miles south of the crossing. How long before the machine and the team will be 15 miles apart measured diagonally across the country?



If t is the number of hours required, then

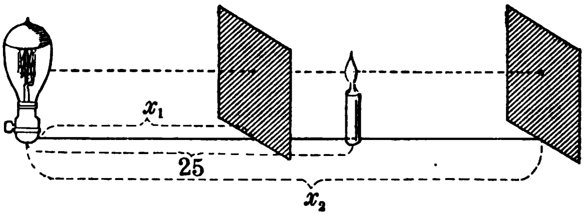
$$(25 - 25\,t)^2 + (10\,t)^2 = 15^2,$$

or

$$25^2(1 - t)^2 + 10^2\,t^2 = 15^2.$$

$$5^2(1 - t)^2 + 2^2\,t^2 = 3^2.$$

23. An electric light of 32 candle power is 25 feet from a candle of 6 candle power. Where should a card be placed between them so as to receive the same amount of light from each?



Suggestion. The illumination varies inversely as the square of the distance from the source of light, and directly as the intensity of the source of light.

Let $x =$ distance from the candle. Then $\frac{6}{x^2} = \frac{32}{(25-x)^2}$. Compute result accurately to two places of decimals.

24. Where must a card be placed in problem 23 if the candle is between the card and the electric light?

Notice that the roots of the equations in 23 and 24 are the same. Explain what this means.

25. If the distance between the earth and the sun is 93 million miles, and if the mass of the sun is 300,000 times that of the earth, find two positions in which a particle would be equally attracted by the earth and the sun.

The gravitational attraction of one body upon another varies inversely as the square of the distance, and directly as the product of the masses. Represent the mass of the earth by unity.

This problem may be solved more easily if we let $d =$ distance from the earth to the sun, and $m =$ the mass of the sun, the mass of the earth being regarded as unity. Let $x =$ the distance from the earth in millions of miles to the point required. The equation, then, is

$$\frac{1}{x^2} = \frac{m}{(d-x)^2}, \text{ or } \frac{1}{x} = \frac{\sqrt{m}}{d-x}.$$

After solving this equation, substitute the numbers given above.

CHAPTER XV

SYSTEMS OF QUADRATIC EQUATIONS

244. One Linear and One Quadratic Equation. A system of equations in two unknowns, one linear and one of the second degree, may always be solved by a substitution, leading to the solution of an ordinary quadratic.

Example 1. Solve the system $\begin{cases} x - y = 4, & (1) \\ 2x^2 + 3y^2 = 84. & (2) \end{cases}$

Solution. From (1), $x = y + 4.$

Substituting in (2), $2(4+y)^2 + 3y^2 = 84,$

or $5y^2 + 16y - 52 = 0.$

Factoring, $(5y + 26)(y - 2) = 0.$

Hence, $y = 2$, or $y = -\frac{26}{5}.$

From $y = 2$ and (1), we have $\begin{cases} x = 6, \\ y = 2. \end{cases}$

From $y = -\frac{26}{5}$ and (1), we have $\begin{cases} x = -\frac{6}{5}, \\ y = -\frac{26}{5}. \end{cases}$

Example 2. Solve the system $\begin{cases} 3x - 7y = 1, & (1) \\ x^2 - 2xy - 2x + y = -3. & (2) \end{cases}$

Solution. From (1), $x = \dfrac{1 + 7y}{3}.$

Substituting in (2),

$$\left(\frac{1+7y}{3}\right)^2 - 2 \cdot \frac{1+7y}{3} y - 2 \cdot \frac{1+7y}{3} + y = -3.$$

Clearing of fractions,

$$1 + 14y + 49y^2 - 6y - 42y^2 - 6 - 42y + 9y = -27,$$

collecting terms $7y^2 - 25y + 22 = 0.$

Factoring, $(y - 2)(7y - 11) = 0$, and $y = 2$, or $\frac{11}{7}.$

From $y = 2$, we have $\begin{cases} x = 5, \\ y = 2. \end{cases}$

From $y = \frac{11}{7}$, we have $\begin{cases} x = 4, \\ y = \frac{11}{7}. \end{cases}$

Example 3. Solve the system $\begin{cases} 2x + 3y = 6, & (1) \\ 4x^2 - 3y^2 - x = 25. & (2) \end{cases}$

From (1),
$$x = \frac{6 - 3y}{2}.$$

Substituting in (2),
$$4 \cdot \left(\frac{6 - 3y}{2}\right)^2 - 3y^2 - \frac{6 - 3y}{2} = 25.$$

Hence,
$$36 - 36y + 9y^2 - 3y^2 - \frac{6 - 3y}{2} = 25,$$
$$72 - 72y + 18y^2 - 6y^2 - 6 + 3y = 50,$$
$$12y^2 - 69y + 16 = 0.$$

Solving by the formula,
$$y = \frac{69 \pm \sqrt{69^2 - 4 \cdot 12 \cdot 16}}{24} = \frac{69 \pm \sqrt{3993}}{24} = .24 \text{ or } 5.51.$$

Substituting these values of y in (1) and (2), we have
$$\begin{cases} x = 2.64, \\ y = .24, \end{cases} \text{ and } \begin{cases} x = -5.27, \\ y = 5.51. \end{cases}$$

WRITTEN EXERCISES

Solve the following systems of equations:

1. $\begin{cases} x^2 + y^2 = 25, \\ x + y = 7. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 10, \\ x + 2y = 5. \end{cases}$

3. $\begin{cases} x^2 - y^2 = 30, \\ x - 2y = 3. \end{cases}$

4. $\begin{cases} y = 2x^2 - 3x - 4, \\ y - x = 3. \end{cases}$

5. $\begin{cases} y = 2x^2 + 2x - 1, \\ 2x - y = 4. \end{cases}$

6. $\begin{cases} y = 3x + 3x^2 - 6, \\ 2y + 3x - 3 = 0. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 40, \\ x + 2y = 10. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 29, \\ 3x - 7y = -29. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 4, \\ ax + 3y = 16. \end{cases}$

10. Solve example 9 if $a = 8$.

11. $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1, \\ 3x + 4y = 12. \end{cases}$

12. $\begin{cases} 3x^2 + 2y^2 = 11, \\ x - 3y = 7. \end{cases}$

SPECIAL METHODS OF SOLUTION

245. Certain systems of one linear and one second degree equation may be solved by special methods, as is shown in the following examples:

Example 1. Solve $\begin{cases} x^2 + y^2 = 13, & (1) \\ x - y = 1. & (2) \end{cases}$

Solution. Squaring both members of (2) and subtracting from (1),

$$2xy = 12. \quad (3)$$

Adding (1) and (3), $\quad x^2 + 2xy + y^2 = 25. \quad (4)$

Taking square roots, $\quad x + y = \pm 5. \quad (5)$

From (2) and (5), $\begin{cases} x = 3, \\ y = 2, \end{cases}$ and $\begin{cases} x = -2, \\ y = -3. \end{cases}$

Example 2. $\begin{cases} x^2 - y^2 = 5, & (1) \\ x - y = 1. & (2) \end{cases}$

Solution. From (1) $\quad (x - y)(x + y) = 5. \quad (3)$

Dividing the members of (3) by those of (2),

$$x + y = 5. \quad (4)$$

Then (2) and (4) may be solved in the usual way.

The system $\begin{cases} x^2 - y^2 = 7, \\ x + y = 7, \end{cases}$ may be solved in a similar manner.

Example 3. Solve $\begin{cases} x + y = 7, & (1) \\ xy = 12. & (2) \end{cases}$

Solution. Multiplying (2) by 4, and subtracting from the square of (1), we get

$$x^2 - 2xy + y^2 = 1, \quad (3)$$

whence, $\quad x - y = \pm 1. \quad (4)$

Then (1) and (4) are linear equations.

The equations $\begin{cases} x - y = 3, \\ xy = 10, \end{cases}$ may be solved in a similar manner.

WRITTEN EXERCISES

Solve the following systems:

1. $\begin{cases} x+y=10, \\ xy=21. \end{cases}$

2. $\begin{cases} x^2+y^2=34, \\ x-y=2. \end{cases}$

3. $\begin{cases} x^2+y^2=32, \\ x+y=8. \end{cases}$

4. $\begin{cases} x^2+y^2=r^2, \\ x+y=a. \end{cases}$

5. $\begin{cases} x+y=a, \\ xy=b. \end{cases}$

6. $\begin{cases} x^2+y^2=r^2, \\ x-y=a. \end{cases}$

7. $\begin{cases} x^2-y^2=a, \\ x+y=b. \end{cases}$

8. $\begin{cases} x^2-y^2=p, \\ x-y=q. \end{cases}$

246. Simultaneous Equations, Each of the Second Degree. We now proceed to study the solution of a pair of equations each of the second degree.

Consider
$$\begin{cases} x^2+y=7, & (1) \\ x+y^2=13. & (2) \end{cases}$$

Solving (1) for y, and substituting in (2), we have
$$x+49-14x^2+x^4=13,$$
which is of the *fourth degree* and cannot be solved by any methods thus far studied. There are, however, special cases in which two equations each of the second degree can be solved by a proper combination of methods already known.

Case I. *When only the squares of the unknowns enter the equations.*

Example. Solve
$$\begin{cases} 2x^2+3y^2=21, & (1) \\ 3x^2-2y^2=25. & (2) \end{cases}$$

Multiplying (1) by 2, and (2) by 3, and adding, we have
$$13x^2=117.$$
$$x^2=9.$$
$$x=\pm 3.$$

Substituting in (1), we have two values of y for each value of x, thus
$$\begin{cases} x=3, \\ y=1; \end{cases} \quad \begin{cases} x=3, \\ y=-1; \end{cases} \quad \begin{cases} x=-3, \\ y=1; \end{cases} \quad \begin{cases} x=-3, \\ y=-1. \end{cases}$$

Case II. ***When all terms containing the unknowns are of the second degree in the unknowns.***

Example. Solve $\begin{cases} 2x^2 - 3xy + 4y^2 = 3, & (1) \\ 3x^2 - 4xy + 3y^2 = 2. & (2) \end{cases}$

First Solution. Put $y = vx$ in (1) and (2), obtaining

$$\begin{cases} x^2(2 - 3v + 4v^2) = 3, & (3) \\ x^2(3 - 4v + 3v^2) = 2. & (4) \end{cases}$$

Hence, from (3) and (4),

$$x^2 = \frac{3}{2 - 3v + 4v^2}, \text{ and also } x^2 = \frac{2}{3 - 4v + 3v^2}. \quad (5)$$

From (5) $$\frac{3}{2 - 3v + 4v^2} = \frac{2}{3 - 4v + 3v^2}, \quad (6)$$

or $$v^2 - 6v + 5 = 0. \quad (7)$$

Hence, $$v = 1, \text{ and } v = 5. \quad (8)$$

From $y = vx$, $$y = x, \text{ and } y = 5x. \quad (9)$$

If $y = x$, then from (1) and (2),

$$\begin{cases} x = 1, \\ y = 1, \end{cases} \text{ and } \begin{cases} x = -1, \\ y = -1. \end{cases}$$

If $y = 5x$, then from (1) and (2),

$$\begin{cases} x = \dfrac{1}{\sqrt{29}}, \\ y = \dfrac{5}{\sqrt{29}}, \end{cases} \text{ and } \begin{cases} x = -\dfrac{1}{\sqrt{29}}, \\ y = -\dfrac{5}{\sqrt{29}}. \end{cases}$$

Verify each of these four solutions by substituting in equations (1) and (2).

Second Solution. We may eliminate the *known terms* by multiplying equation (1) by 2 and equation (2) by 3, and subtracting,

thus, $$5x^2 - 6xy + y^2 = 0.$$

Dividing by x^2,

$$5 - 6\frac{y}{x} + \frac{y^2}{x^2} = 0.$$

Factoring, $$\left(\frac{y}{x} - 5\right)\left(\frac{y}{x} - 1\right) = 0.$$

Hence, $\dfrac{y}{x} = 5$ or 1, and $y = 5x$, or x.

These are the results reached in equation (9) of the first solution, and the remainder of the work is the same as before.

WRITTEN EXERCISES

Solve the following systems of equations:

1. $\begin{cases} 2x^2 + 4y^2 = 38, \\ 5x^2 - y^2 = -4. \end{cases}$

2. $\begin{cases} 3x^2 - 2y^2 = 97, \\ x^2 + y^2 = 74. \end{cases}$

3. $\begin{cases} x^2 - y^2 = 27, \\ 2x^2 - 3y^2 = 45. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 20, \\ 2x^2 - y^2 = 7. \end{cases}$

5. $\begin{cases} x^2 - y^2 = 10, \\ x^2 + 3y^2 = 34. \end{cases}$

6. $\begin{cases} 3x^2 - 2y^2 = 40, \\ 4x^2 - 3y^2 = 60. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 50, \\ x^2 - y^2 = 14. \end{cases}$

8. $\begin{cases} 3x^2 + y^2 = 14, \\ 2x^2 - y^2 = 6. \end{cases}$

9. $\begin{cases} 5x^2 + y^2 = 42, \\ x^2 + 5y^2 = 18. \end{cases}$

10. $\begin{cases} x^2 + y^2 = a, \\ x^2 - y^2 = b. \end{cases}$

11. $\begin{cases} 2x^2 + 5y^2 = a, \\ 3x^2 + y^2 = b. \end{cases}$

12. $\begin{cases} 7x^2 - 2y^2 = a, \\ 3x^2 - y^2 = b. \end{cases}$

13. $\begin{cases} x^2 + y^2 = a, \\ x^2 - xy + y^2 = b. \end{cases}$

14. $\begin{cases} 2x^2 + 3y^2 = c, \\ 3x^2 + 2y^2 = d. \end{cases}$

15. $\begin{cases} ax^2 + by^2 = h, \\ cx^2 + dy^2 = k. \end{cases}$

16. $\begin{cases} x^2 + xy + y^2 = 3, \\ x^2 + 3xy - 2y^2 = 2. \end{cases}$

17. $\begin{cases} 2x^2 - xy + y^2 = 7, \\ x^2 + 2xy - y^2 = 7. \end{cases}$

18. $\begin{cases} 3x^2 - 4xy + y^2 = -1, \\ x^2 + 2xy - 3y^2 = -7. \end{cases}$

19. $\begin{cases} 5x^2 - 7xy + 3y^2 = 1, \\ 4x^2 + 5xy - 8y^2 = 1. \end{cases}$

20. $\begin{cases} 3x^2 - 7xy - y^2 = 9, \\ 2x^2 + 4xy - 3y^2 = -5 \end{cases}$

21. $\begin{cases} x^2 - 2xy - y^2 = -14, \\ 2x^2 + 5xy + y^2 = 26. \end{cases}$

22. $\begin{cases} x^2 + y^2 + xy = 3, \\ 2x^2 - y^2 - xy = 9. \end{cases}$

23. $\begin{cases} 4xy + 3x^2 + 2y^2 = 9, \\ 2xy + x^2 + y^2 = 0. \end{cases}$

24. $\begin{cases} 7x^2 + 4xy - y^2 = 8, \\ 3x^2 - 2xy - 4y^2 = 4. \end{cases}$

247. Special Cases. There are many other special cases of simultaneous equations which can be solved by proper combination of the methods thus far used.

The suggestions given in the following examples illustrate the devices in most common use.

The solution should in each case be completed by the student.

Example 1. Solve
$$\begin{cases} x^2 + y^2 = 58, & (1) \\ xy = 21. & (2) \end{cases}$$

Adding twice (2) to (1) and taking square roots, we have

$$x + y = 10, \text{ and } x + y = -10. \quad (3)$$

Each of the equations (3) may now be solved simultaneously with (2), as in Example 3, page 194.

Example 2. Solve
$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5, & (1) \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = 13. & (2) \end{cases}$$

Let $\dfrac{1}{x} = a$ and $\dfrac{1}{y} = b$. Then these equations reduce to

$$\begin{cases} a + b = 5, & (3) \\ a^2 + b^2 = 13. & (4) \end{cases}$$

Solving as in Example 1, page 194, $\begin{cases} a = 3, \\ b = 2, \end{cases}$ $\begin{cases} a = 2, \\ b = 3. \end{cases}$

Hence, $\begin{cases} x = \frac{1}{3}, \\ y = \frac{1}{2}, \end{cases}$ $\begin{cases} x = \frac{1}{2}, \\ y = \frac{1}{3}. \end{cases}$

Example 3. Solve
$$\begin{cases} x^2 + y^2 + x + y = 8, & (1) \\ xy = 2. & (2) \end{cases}$$

Add twice (2) to (1), obtaining

$$x^2 + 2\,xy + y^2 + x + y = 12. \quad (3)$$

Let $x + y = a$. Then (3) reduces to

$$a^2 + a = 12,$$

or
$$a = 3,\ a = -4. \quad (4)$$

Hence, $$x + y = 3, \text{ and } x + y = -4. \quad (5)$$

Now solve each equation in (5) simultaneously with (2).

Example 4. Solve $\begin{cases} x^4y^4 + x^2y^2 = 272, & (1) \\ x^2 + y^2 = 10. & (2) \end{cases}$

In (1) substitute a for x^2y^2. Then

$$a^2 + a = 272, \text{ whence } a = 16, \text{ and } -17.$$

Hence, $$xy = \pm\sqrt{16} = \pm 4, \text{ and } \pm\sqrt{-17}.$$

Each of these four equations may now be solved simultaneously with (2), as in Example 1, page 198.

Example 5. Solve $\begin{cases} x^3 - y^3 = 117, & (1) \\ x - y = 3, & (2) \end{cases}$

By factoring, (1) becomes

$$(x - y)(x^2 + xy + y^2) = 117. \quad (3)$$

Substituting 3 for $x - y$, we have

$$x^2 + xy + y^2 = 39. \quad (4)$$

(2) and (4) may now be solved by substitution as in § 244.

Example 6. Solve $\begin{cases} x^3 + y^3 = 513, & (1) \\ x + y = 9. & (2) \end{cases}$

Factor (1) and substitute 9 for $x + y$. Then proceed as in Example 5.

Example 7. Solve $\begin{cases} x^2y + xy^2 = 126, & (1) \\ x + y = 9. & (2) \end{cases}$

Factoring (1) and substituting 9 for $x + y$, we have

$$xy = 14. \quad (3)$$

(2) and (3) may then be solved as in Example 3, page 194.

Example 8. Solve $\begin{cases} x^3 + y^3 = 54\,xy, & (1) \\ x + y = 6. & (2) \end{cases}$

Factor (1) and substitute 6 for $x + y$, obtaining

$$x^2 - xy + y^2 = 9\,xy. \quad (3)$$

(2) and (3) may now be solved by substitution, as in § 244.

Example 9. Solve $\begin{cases} x^3 + y^3 = 243, & (1) \\ x^2y + xy^2 = 162. & (2) \end{cases}$

Multiply (2) by 3 and add to (1), obtaining a perfect cube.
Taking cube roots, we have

$$x + y = 9. \quad (3)$$

(1) and (3) are now solved as in Example 6 above.

WRITTEN EXERCISES

Solve each of the following pairs of equations:

1. $\begin{cases} r^2 + rs + s^2 = 63, \\ r - s = 3. \end{cases}$

2. $\begin{cases} 3x^2 + 2y^2 = 35, \\ 2x^2 - 3y^2 = 6. \end{cases}$

3. $\begin{cases} 3x^2 + 2xy = 16, \\ 4x^2 - 3xy = 10. \end{cases}$

4. $\begin{cases} a^2 + ab + b^2 = 7, \\ a^2 - ab + b^2 = 19. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 5, \\ xy = 2. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 5, \\ x^2 + z^2 = 10, \\ y^2 + z^2 = 13. \end{cases}$

7. $\begin{cases} 3x - 4y = 0, \\ x^2 + y^2 = 4. \end{cases}$

8. $\begin{cases} x^2 + xy = 4, \\ y^2 + xy = 5. \end{cases}$

9. $\begin{cases} x^3 + y^3 = 91, \\ x + y = 7. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 11, \\ x^2 - y^2 = 7. \end{cases}$

11. $\begin{cases} x^2 - 3xy = 0, \\ 5x^2 + 3y^2 = 9. \end{cases}$

12. $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = 19, \\ \dfrac{1}{x} + \dfrac{1}{y} = 1. \end{cases}$

13. $\begin{cases} 3x - 2y = 6, \\ 3x^2 - 2xy + 4y^2 = 12. \end{cases}$

14. $\begin{cases} a + b + ab = 11, \\ (a + b)^2 + a^2b^2 = 61. \end{cases}$

15. $\begin{cases} \dfrac{1}{a^2} + \dfrac{1}{ab} + \dfrac{1}{b^2} = 49, \\ \dfrac{1}{a} + \dfrac{1}{b} = 8. \end{cases}$

Suggestions. In example 12, substitute $\dfrac{1}{x} = a, \dfrac{1}{y} = b.$

In 14, $a + b = x,\ ab = y.$

In 15, $\dfrac{1}{a} = x,\ \dfrac{1}{b} = y.$

16. $\begin{cases} 4a^2 - 2ab = b^2 - 16, \\ 5a^2 = 7ab - 36. \end{cases}$

17. $\begin{cases} 3x^2 - 9y^2 = 12, \\ 2x - 3y = 14. \end{cases}$

18. $\begin{cases} x^2 + xy + y^2 = 19, \\ x^2 + y^2 = 13. \end{cases}$

19. $\begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$

20. $\begin{cases} x^2 + y^2 + x - y = 36, \\ xy = 15. \end{cases}$

21. $\begin{cases} x^2 - 5xy + y^2 = -2, \\ x^2 + 7xy + y^2 = 22. \end{cases}$

22. $\begin{cases} a^2 + 6ab + b^2 = 124, \\ a + b = 8. \end{cases}$

23. $\begin{cases} a^2 - 3ab + 2b^2 = 0, \\ 2a^2 + ab - b^2 = 9. \end{cases}$

24. $\begin{cases} x^2+y^2+2x+2y=27, \\ xy=-12. \end{cases}$

25. $\begin{cases} x^2+y^2-5x-5y=-4, \\ xy=5. \end{cases}$

26. $\begin{cases} (7+x)(6+y)=80, \\ x+y=5. \end{cases}$

27. $\begin{cases} (x-4)^2+(y+4)^2=100, \\ x+y=14. \end{cases}$

28. $\begin{cases} b+a^2=5(a-b), \\ a+b^2=2(a-b). \end{cases}$

29. $\begin{cases} (13x)^2+2y^2=177, \\ (2y)^2-13x^2=3. \end{cases}$

30. $\begin{cases} x^2+y^2=20, \\ 5x^2-3y^2=28. \end{cases}$

31. $\begin{cases} x^2=-5-3xy, \\ 2xy=y^2-24. \end{cases}$

32. $\begin{cases} x+y+\sqrt{x+y}=12, \\ x^3+y^3=189. \end{cases}$

33. $\begin{cases} x^4+x^2y^2+y^4=133, \\ x^2-xy+y^2=7. \end{cases}$

34. $\begin{cases} x+xy+y=29, \\ x^2+xy+y^2=61. \end{cases}$

35. $\begin{cases} 2x^2-5xy+3x-2y=22, \\ 5xy+7x-8y-2x^2=8. \end{cases}$

36. $\begin{cases} x+y=74, \\ x^2+y^2=3026. \end{cases}$

37. $\begin{cases} x^3+y^3=9, \\ x+y=3. \end{cases}$

38. $\begin{cases} x^3-y^3=37, \\ x-y=1. \end{cases}$

39. $\begin{cases} x^2+y^2-xy=80, \\ x-y-xy=-8. \end{cases}$

40. $\begin{cases} 7y^2-5x^2+20x+13y=29, \\ 5(x-2)^2-7y^2-17y=-17. \end{cases}$

41. $\begin{cases} 8a+8b-ab-a^2=18, \\ 5a+5b-b^2-ab=24. \end{cases}$

42. $\begin{cases} 2(x+4)^2-5(y-7)^2=75, \\ 7(x+4)^2+15(y-7)^2=1075. \end{cases}$

43. $\begin{cases} x^3+y^3=(a+b)(x-y), \\ x^2-xy+y^2=a-b. \end{cases}$

44. $\begin{cases} x+y=4, \\ x^3+x^2y+xy^2+y^3=32. \end{cases}$

45. $\begin{cases} (3x+4y)(7x-2y)+3x+4y=44, \\ (3x+4y)(7x-2y)-7x+2y=30. \end{cases}$

PROBLEMS IN TWO OR MORE UNKNOWNS LEADING TO QUADRATICS

248. Interpretation of Results. In solving a problem leading to a system of quadratic equations, it may be that not all pairs of values resulting from the equations will satisfy the conditions of the problem. See § 243.

In each case it is necessary to determine whether all or any of the results are applicable to the problem, as in the following example.

Example. The sum of the length and width of a rectangle is 18, and its area is 90. Find the dimensions.

Solution. Let the dimensions be x and y.

Then $$\begin{cases} x + y = 18, \\ xy = 90. \end{cases}$$

Solving, $$\begin{cases} x = 9 + 3\sqrt{-1}, \\ y = 9 - 3\sqrt{-1}, \end{cases} \quad \begin{cases} x = 9 - 3\sqrt{-1}, \\ y = 9 + 3\sqrt{-1}. \end{cases}$$

The imaginary results show that the problem has no real solution. That is, there is no rectangle having the properties stated in the problem.

PROBLEMS

Determine in each case whether all results are applicable to the problem.

1. The area of a rectangle is 2400 square feet and its perimeter is 200 feet. Find the length of its sides.

2. The area of a rectangle is a square feet and its perimeter is $2\,b$ feet. Find the length of its sides. Solve Example 1 by substituting in the formula thus obtained.

3. The sides a and b of a right triangle are increased by the same amount, thereby increasing the square on the hypotenuse by $2\,k$. Find by how much each side is increased.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

4. The hypotenuse c and one side a of a right triangle are each increased by the same amount, thereby increasing the square on the other side by $2k$. Find how much was added to the hypotenuse.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

5. Find the altitude drawn to the longest side of the triangle whose sides are 6, 7, 8.

6. Find the area of a triangle whose sides are 15, 17, 20.

First find one altitude as in problem 5.

7. Find the area of a triangle whose base is 16 and whose sides are 10 and 14.

8. Find the altitude on a side a of a triangle two of whose sides are a and a third b.

9. The sum of two numbers less 2, divided by their difference is 4, and the sum of their cubes divided by the difference of their squares is $1\frac{2}{5}$ times their sum. Find the numbers.

10. In going one mile the front wheel of a carriage makes 88 revolutions more than the rear wheel. If one foot is added to the circumference of the rear wheel, and 3 feet to that of the front wheel, the latter will make 22 revolutions more than the former. Find the circumference of each wheel.

11. The circumference of the rear wheel of a carriage is a feet greater than that of the front wheel. In running d feet the front wheel makes n more revolutions than the rear wheel. Find the circumference of each wheel.

12. Find a number of two digits whose sum divided by their difference is 4. This number divided by the sum of its digits is equal to twice the digit in units' place plus $\frac{1}{8}$ of the digit in tens' place; the tens' digit being the greater.

13. Find a simple fraction such that, if 3 is added to each of its terms, the result is $\frac{4}{5}$, and if 3 is subtracted from each of its terms, the result is $\frac{1}{10}$ of its numerator.

14. Find a simple fraction such that, if 2 is added to each of its terms, its value is $\frac{5}{6}$, and if 2 is subtracted from each of its terms, its value is $\frac{1}{8}$ of its denominator.

Note. In Exs. 13 and 14 find also a *complex* fraction as the answer.

15. The diagonal of a rectangle is a and its perimeter $2b$. Find its sides.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

16. If the difference between the length and width of a rectangle is b and the diagonal is a, find the sides.

17. The hypotenuse of a right triangle is 20 inches longer than one side and 10 inches longer than the other. Find the dimensions of the triangle.

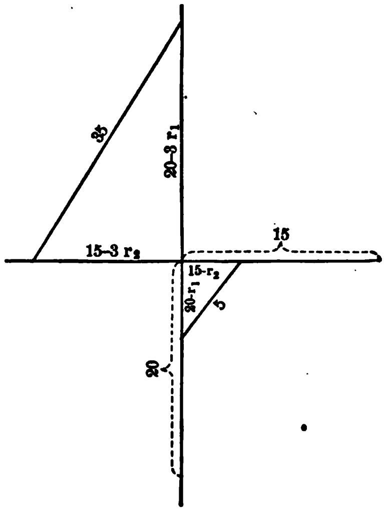
18. If in problem 17 the hypotenuse is a inches longer than one side and b inches longer than the other, find the dimensions of the triangle.

19. The area of a circle exceeds that of a square by 10 square inches, while the perimeter of the circle is 4 less than that of the square. Find the side of the square and the radius of the circle.

Use $3\frac{1}{7}$ as the value of π.

20. The upper base of a trapezoid is equal to the altitude, and the area is 48. If the altitude is decreased by 4, and the upper base by 2, the area is then 14. Find the dimensions of the trapezoid.

21. An automobile running northward at the rate of 15 miles per hour is 20 miles south of the intersection with an east and west road. At the same time another automobile running westward on the cross-road at the rate of 20 miles per hour is 15 miles east of the crossing. How far apart (diagonally) will they be 15 minutes later? One hour later?



22. Under the conditions of problem 21 how long after the first time specified will the automobiles be 10 miles apart? Is there more than one such position?

23. What are the rates of motion of the automobiles in problem 21 if one hour later they are 5 miles apart and 3 hours later they are 35 miles apart? (See the figure.)

If the rates of the automobiles are r_1 and r_2, then after 1 hour we have

$$\begin{cases} (20 - r_1)^2 + (15 - r_2)^2 = 5^2, & (1) \\ (20 - 3\,r_1)^2 + (15 - 3\,r_2)^2 = 35^2. & (2) \end{cases}$$

and after 3 hours we have

Simplifying (1) and (2), $r_1^2 + r_2^2 - 40\,r_1 - 30\,r_2 + 600 = 0$, (3)

and $9\,r_1^2 + 9\,r_2^2 - 120\,r_1 - 90\,r_2 - 600 = 0.$ (4)

Multiplying (3) by 9, subtracting from (4), and simplifying,

$$4\,r_1 + 3\,r_2 = 100. \quad (5)$$

The solution of this problem may now be completed by solving (5) and (1) simultaneously.

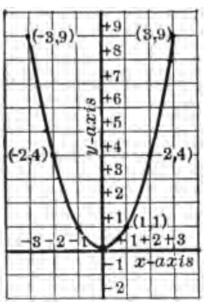
NOTE. — In equation (2), $20 - 3\,r_1$ and $15 - 3\,r_2$ are both negative numbers. This means that in this problem the distances north and west of the crossing are *negative*, while those south and east are *positive*.

CHAPTER XVI

GRAPHIC REPRESENTATION OF QUADRATICS

249. Graphs of Quadratics. We saw, § 106, that a single equation in two variables is satisfied by indefinitely many pairs of numbers. If such an equation is of the **first degree** in the two variables, the graph is in every case a *straight line.*

We are now to consider graphs of equations of the **second degree** in two variables.



Example 1. Graph the equation $y = x^2$.

By giving various values to x and computing the corresponding value of y, we find pairs of numbers as follows which satisfy this equation:

$$\begin{cases} x = 0, \\ y = 0. \end{cases} \quad \begin{cases} x = 1, \\ y = 1. \end{cases} \quad \begin{cases} x = -1, \\ y = 1. \end{cases}$$

$$\begin{cases} x = 2, \\ y = 4. \end{cases} \quad \begin{cases} x = -2, \\ y = 4. \end{cases} \quad \begin{cases} x = 3, \\ y = 9. \end{cases}$$

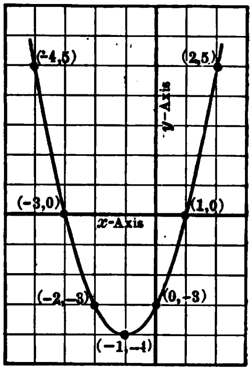
$$\begin{cases} x = -3, \\ y = 9. \end{cases} \text{ etc.}$$

These pairs of numbers correspond to points which lie on a curve as shown in the figure.

By referring to the graph the curve is seen to be symmetrical with respect to the y-axis. This can be seen directly from the equation itself, since x is involved only as a square, and hence, if $y = x^2$ is satisfied by $x = a$, $y = b$, it must also be satisfied by $x = -a$, $y = b$.

It may easily be verified that no three points of this curve lie on a straight line. The curve is called a **parabola.**

Example 2. Graph the equation $y = x^2 + 2x - 3$.



Each of the following pairs of numbers satisfies the equation:

$$\begin{cases} x = 0, \\ y = -3. \end{cases} \begin{cases} x = 1, \\ y = 0. \end{cases} \begin{cases} x = -1, \\ y = -4. \end{cases}$$

$$\begin{cases} x = 2, \\ y = 5. \end{cases} \begin{cases} x = -2, \\ y = -3. \end{cases} \begin{cases} x = -3, \\ y = 0. \end{cases}$$

$$\begin{cases} x = -4, \\ y = 5. \end{cases}$$

Plotting these points and drawing a smooth curve through them, we have the graph of the equation, as shown in the figure.

WRITTEN EXERCISES

In this manner graph each of the following:

1. $y = x^2 - 1$.
2. $y = x^2 + 4x$.
3. $y = x^2 + 3x - 4$.
4. $y = x^2 + 5x + 4$.
5. $y = x^2 - 7x + 6$.
6. $y = 3x^2 - 7x + 2$.

250. Intersection Points on the x-axis. We now seek to find the points at which each of the above curves cuts the x-axis. The value of y for all points on the x-axis is zero. Hence we put $y = 0$, and solve the resulting equation.

Thus in Example 2 above, if $y = 0$, $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$, which is satisfied by $x = 1$ and $x = -3$. Hence this curve cuts the x-axis in the two points $x = 1$, $y = 0$ and $x = -3$, $y = 0$, as shown in the figure.

Similarly in Example 1 on page 206, if $y = 0$, $x^2 = 0$, and hence $x = 0$. Hence the curve meets the x-axis in the point $x = 0$, $y = 0$, as shown in the figure.

Again, in $y = 2x^2 - x - 6$, if $y = 0$ then $2x^2 - x - 6 = (2x + 3)(x - 2) = 0$, which is satisfied by $x = -\frac{3}{2}$ and $x = 2$. Hence the curve cuts the x-axis in the points $x = -\frac{3}{2}$ and $x = 2$.

WRITTEN EXERCISES

Find the points in which each of the six curves in the preceding list cuts the x-axis.

Notice that in every case the expression to the right of the equality sign can be factored, so that when $y = 0$ the resulting equation in x may be solved by the factoring method.

Example 3. Plot the curve $y = x^2 + 4x + 2$ and find its intersection points with the x-axis.

We are not able to factor $x^2 + 4x + 2$ by inspection. Hence, we solve the equation $x^2 + 4x + 2 = 0$ by completing the square, or by the formula, obtaining $x = -2 + \sqrt{2}$ and $x = -2 - \sqrt{2}$. Hence, the curve cuts the x-axis in points whose abscissas are given by these irrational values of x.

NOTE. — In making this graph, we first plot points corresponding to *integral* values of x, as before; then, in drawing the smooth curve through these, the intersections made with the x-axis are approximately the points on the number scale corresponding to the irrational numbers, $-2 + \sqrt{2}$ and $-2 - \sqrt{2}$.

WRITTEN EXERCISES

In the above manner, find the points at which each of the following curves cuts the x-axis, and plot the curves:

1. $y = x^2 + 5x + 3.$
2. $y = 3x^2 + 8x - 2.$
3. $y = 6x - 4x^2 + 5.$
4. $y = -4 - 2x + 5x^2.$
5. $y = 2x - 5x^2 + 8.$
6. $y = 5 + 8x - 3x^2.$

251. Intersection Points on Lines Parallel to the x-axis. The points in which a line parallel to the x-axis cuts a curve may be found in a manner shown by the following examples.

Example 1. Graph on the same axes the straight line, $y = -2$ and the curve, $y = x^2 + 2x - 3$.

The line $y = -2$ is parallel to the x-axis and two units *below* it. If we substitute -2 for y in the equation

$$y = x^2 + 2x - 3,$$

and then solve for x, we shall find the abscissas of the two points in which the line $y = -2$ cuts this curve.

Solving, $\quad -2 = x^2 + 2x - 3$ for x,

we find $\quad x_1 = -1 + \sqrt{2}$ and $x_2 = -1 - \sqrt{2}$.

Hence, the line $y = -2$ cuts the curve $y = x^2 + 2x - 3$ in the points:

$$\begin{cases} x_1 = -1 + \sqrt{2}, \\ y_1 = -2, \end{cases} \quad \text{and} \quad \begin{cases} x_2 = -1 - \sqrt{2}, \\ y_2 = -2. \end{cases}$$

Example 2. Graph on the same axes $y = -4$ and $y = x^2 + 2x - 3$.

Substituting and solving as before, we find

$$x_1 = \frac{-2 + \sqrt{4 - 4}}{2} = \frac{-2 + 0}{2} = -1;$$

and

$$x_2 = \frac{-2 - \sqrt{4 - 4}}{2} = \frac{-2 - 0}{2} = -1.$$

In this case the two values of x are *equal*, and hence there is only *one* point common to the line and the curve, namely, the point $x = -1$, $y = -4$. This may be understood by thinking of the line $y = -2$, in the preceding example, as moving down to the position $y = -4$, whereupon the two values of x which were *distinct* come to *coincide*.

252. Tangent to a Curve. A line which meets a curve in two *coincident points* is said to be *tangent to the curve.*

253. Problem. To find a Line Tangent to a Curve. What is the value of a in $y = a$, if this line is tangent to the curve $y = x^2 + 5x + 8$?

Substituting a for y and solving by means of the formula, we have

$$x = \frac{-5 \pm \sqrt{25 - 4(8 - a)}}{2} = \frac{-5 \pm \sqrt{4a - 7}}{2}.$$

If the line is to be tangent to the curve, then the expression under the radical sign must be zero. That is, $4a - 7$, or $a = \frac{7}{4}$. Verify by plotting the graphs.

WRITTEN EXERCISES

In each of the following find the value of a in order that $y = a$ may be tangent to the curve:

1. $y = x^2 - 1$.
2. $y = x^2 + 4x$.
3. $y = x^2 + 3x - 4$.
4. $y = x^2 + 5x + 4$.
5. $y = x^2 - 7x + 16$.
6. $y = 3x^2 - 7x + 2$.

254. Problem. To find a Line which does not cut a Given Curve. Find the intersection points of the curve $y = x^2 + 3x + 5$ and the line $y = 2\frac{1}{2}$.

Substituting for y and solving for x, we have

$$x_1 = \frac{-6 + \sqrt{36 - 40}}{4} = \frac{-6 + 2\sqrt{-1}}{4} = \frac{-3 + \sqrt{-1}}{2};$$

$$x_2 = \frac{-6 - \sqrt{36 - 40}}{4} = \frac{-6 - 2\sqrt{-1}}{4} = \frac{-3 - \sqrt{-1}}{2}.$$

The imaginary solutions indicate that the curve and the line have *no point in common*, as is evident on constructing the graphs.

If now we solve the equations $y = a$ and $y = x^2 + 3x + 5$ for x by first substituting a for y, we have

$$x = \frac{-3 \pm \sqrt{4a - 11}}{2}.$$

We now see that

(1) If $a > \frac{11}{4}$, the number under the radical sign is *positive*, and there are *two real and distinct* values of x. Hence the line $y = a$ and the curve meet in two points.

(2) If $a < \frac{11}{4}$, the number under the radical sign is *negative*. Consequently the values of x are *imaginary* and the line $y = a$ and the curve do not meet.

(3) If $a = \frac{11}{4}$, the number under the radical sign is zero, and the two values of x are *equal*. Hence the line is *tangent* to the curve.

255. Nature of Roots Determined by Formula. From the two preceding paragraphs it appears that it is possible to determine the *relative* positions of the line and the curve *without completely solving* the equations. Namely, eliminate y and reduce the resulting equation to the form $ax^2 + bx + c = 0$. Hence we have:

(1) If $b^2 - 4ac > 0$, *i.e. positive*, then the line cuts the curve in two distinct points.

(2) If $b^2 - 4ac = 0$, then the line is tangent to the curve.

(3) If $b^2 - 4ac < 0$, *i.e. negative*, then the line does not cut the curve. See § 240.

Example 1. Does the line $y = 2x - 6$ meet the curve $y = x^2 + 7x - 4$?

Solution. Substituting from the first equation in the second,

$$2x - 6 = x^2 + 7x - 4, \text{ or } x^2 + 5x + 2 = 0.$$

Then $b^2 - 4ac = 25 - 8 = 17$. Hence the line meets the curve in two distinct points.

Example 2. Does the line $x + y = 3$ meet the curve $y = x^2 - 3x + 4$?

Solution. Substituting from the first equation in the second,

$$3 - x = x^2 - 3x + 4 \text{ or } x^2 - 2x + 1 = 0.$$

Then $b^2 - 4ac = 4 - 4 = 0$. Hence the two roots are equal and the line meets the curve in only one point and is tangent to it.

Example 3. Does the line $2x - 3y = 10$ meet the curve $y = x^2 + 4x + 1$?

Solution. Eliminating y,

$$2x - 10 = 3x^2 + 12x + 3 \text{ or } 3x^2 + 10x + 13 = 0.$$

Then $b^2 - 4ac = 100 - 156 = -56$ and the line does not meet the curve.

WRITTEN EXERCISES

In each of the following determine without graphing whether or not the line meets the curve, and in case it does, find the intersection points:

1. $\begin{cases} y = 2x^2 - 3x - 4, \\ y - x = 3. \end{cases}$

2. $\begin{cases} y = 2x^2 + 2x - 1, \\ 2y = x - 1. \end{cases}$

3. $\begin{cases} y = 3x^2 - 2x - 1, \\ 2x - y = 4. \end{cases}$

4. $\begin{cases} y = 4x^2 + 6x + 1, \\ x = y + 5. \end{cases}$

5. $\begin{cases} y = x^2 - 7x + 12, \\ 5x - y = -1. \end{cases}$

6. $\begin{cases} y = 5x^2 + 8x + 3, \\ 2y - 5x - 2 = 0. \end{cases}$

7. $\begin{cases} y = 5x^2 - 8x + 3, \\ 3 - x = 3y. \end{cases}$

8. $\begin{cases} y = -5x^2 + 8x - 3, \\ 2 - 4y - x = 0. \end{cases}$

9. $\begin{cases} y = -5x^2 - 8x - 3, \\ 5y - 3x = 8. \end{cases}$

10. $\begin{cases} y = 3x - 3x^2 + 7, \\ -5 - 3x + 2y = 0. \end{cases}$

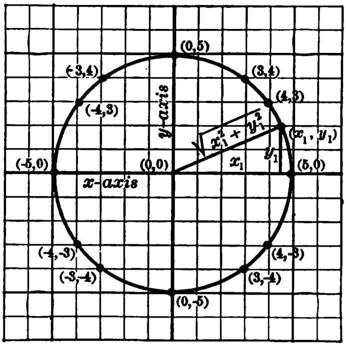
256. The Equation of the Circle. Graph the equation

$$x^2 + y^2 = 25.$$

Writing the equation in the form $y = \pm\sqrt{25 - x^2}$, and assigning values to x, we compute the corresponding values of y as follows:

$$\begin{cases} x = 0, \\ y = \pm 5. \end{cases} \quad \begin{cases} x = \pm 5, \\ y = 0. \end{cases} \quad \begin{cases} x = 3, \\ y = \pm 4. \end{cases}$$

$$\begin{cases} x = -3, \\ y = \pm 4. \end{cases} \quad \begin{cases} x = 4, \\ y = \pm 3. \end{cases} \quad \begin{cases} x = -4, \\ y = \pm 3. \end{cases}$$



These points lie on the circumference of a circle whose radius is 5. For, if we consider any point (x_1, y_1) on this circumference, then we have $x_1^2 + y_1^2 = 25$, since the sum of the squares on the sides of a right triangle is equal to the square on the hypotenuse.

$x^2 + y^2 = 25$ is, therefore, the equation of a circle with radius 5.

257. Problem. To find the points of intersection of the circle $x^2 + y^2 = 25$ and the line $x + y = 7$.

Solving these equations simultaneously we have, $\begin{cases} x_1 = 4, \\ y_1 = 3, \end{cases}$ and $\begin{cases} x_2 = 3, \\ y_2 = 4, \end{cases}$ which represent the required points.

Verify this by graphing the two equations on the same axes.

HISTORICAL NOTE

The **graph** was brought into mathematics by **Descartes**. On the one hand the graph has helped greatly in acquiring a full understanding of the meaning of equations and their solutions; and on the other hand, it has made possible the study of geometric curves by means of the methods of algebra and higher branches of mathematics such as the calculus.

Newton proved that a body moving around the sun under the law of gravitation must always travel in a curve which may be represented by an equation of the second degree. That is, the path is a *circle*, an *ellipse*, a *parabola*, or *hyperbola*. All the planets move in elliptical orbits which are very nearly circular. The orbits of many comets are hyperbolas, some of them being very nearly if not quite parabolic. Mathematical astronomy is based entirely on the algebraic representation of these curves.

258. Problem. To find the points of intersection of the circle $x^2+y^2=25$ and the line $3x+4y=25$.

Eliminating y and solving for x, we find $x=\frac{6\pm 0}{2}=3$.

Hence, $\begin{cases} x_1=3, \\ y_1=4, \end{cases}$ and $\begin{cases} x_2=3, \\ y_2=4. \end{cases}$

Since the two values of x coincide, and likewise the two values of y, the circumference and the line have but *one point* in common. Verify by graphing the line and the circle on the same axes.

259. Problem. To find the points of intersection of the Circle,

$$x^2+y^2=25$$

and the straight line,

$$x+y=10.$$

Substituting for y and solving for x, we have

$$x=\frac{20\pm\sqrt{400-600}}{4}=\frac{20\pm\sqrt{-200}}{4}$$

$$=\frac{20\pm 10\sqrt{-2}}{4}=\frac{10\pm 5\sqrt{-2}}{2}.$$

The imaginary values of x indicate that there is no intersection point. Verify by plotting.

EXERCISES

In each of the following determine by solving whether the line and the circumference meet, and in case they do find the points of intersection:

1. $\begin{cases} x^2+y^2=16, \\ x+y=4. \end{cases}$

2. $\begin{cases} x^2+y^2=36, \\ 4x+y=6. \end{cases}$

3. $\begin{cases} x^2+y^2=25, \\ 2x+y=-5. \end{cases}$

4. $\begin{cases} x^2+y^2=20, \\ 2x+y=0. \end{cases}$

5. $\begin{cases} x^2+y^2=7, \\ x+y=8. \end{cases}$

6. $\begin{cases} x^2+y^2=8, \\ x-y=4. \end{cases}$

7. $\begin{cases} x^2+y^2=16, \\ 3x-4y=12. \end{cases}$

8. $\begin{cases} x^2+y^2=16, \\ 2x+y=12. \end{cases}$

9. $\begin{cases} x^2+y^2=9, \\ x+y=8. \end{cases}$

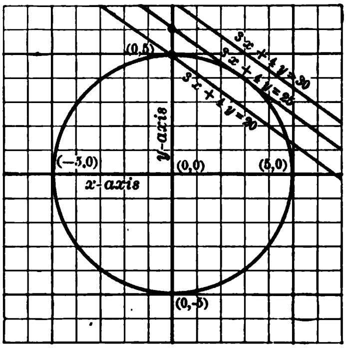
10. $\begin{cases} x^2+y^2=4, \\ x-y=4. \end{cases}$

11. $\begin{cases} x^2+y^2=6, \\ x-2y=6. \end{cases}$

12. $\begin{cases} x^2+y^2=8, \\ 2x-y=8. \end{cases}$

260. Problem. Graph on the same axes the circle $x^2 + y^2 = 5^2$, and the lines $3x + 4y = 20$, $3x + 4y = 25$, and $3x + 4y = 30$.

The first line *cuts* the circumference in two distinct points, the second seems to be *tangent* to it, and the third does *not meet* it. Observe that the three lines are parallel.



In order to make a general discussion of the relative positions of such straight lines and the circumference of a circle, we solve the following equations simultaneously:

$$x^2 + y^2 = r^2, \tag{1}$$
$$3x + 4y = c. \tag{2}$$

Eliminating y by substitution, and solving for x, we find

$$x = \frac{3c \pm 4\sqrt{25r^2 - c^2}}{25}. \tag{3}$$

The two values of x from (3) are the abscissas of the points of intersection of the circumference (1) and the line (2).

(1) These values of x are *real and distinct*, if $25r^2 - c^2$ is *positive*. That is, if $r = 5$, and $c = 20$, then $25r^2 - c^2 = 25 \cdot 25 - 400 = 225$ and the line cuts the circle in two distinct points.

(2) These values of x are real and coincident if $25r^2 - c^2 = 0$. Thus, if $r = 5$, $c = 25$, then $25r^2 - c^2 = 625 - 625 = 0$ and the line meets the circle in only one point. That is, the line is tangent to the circle.

(3) These values of x are imaginary if $25r^2 - c^2$ is negative. Thus, if $r = 5$, $c = 30$, then $25r^2 - c^2 = 625 - 900 = -275$, and the line fails to meet the circle.

Notice that the three lines are parallel to each other and that their equations differ only in their right members. Hence, if in the equation $3x + 4y = c$, the value of c increases from 20 to 25, the line will move outward while remaining parallel to its original position. The points of intersection will gradually approach each other until they coincide. At the same time the expression $25r^2 - c^2$ will gradually approach zero.

261. Parameters. Letters such as c and r in the above solution to which any arbitrary constant values may be assigned are called **parameters**, while x and y are the **unknowns** of the equations.

EXERCISES

Solve each of the following pairs of equations:

Give such values to the parameters involved as will cause the line (a) to cut the curve in two distinct points, (b) to be tangent to the curve, (c) to fail to meet the curve.

1. $\begin{cases} x^2+y^2=4, \\ ax+3y=16. \end{cases}$

2. $\begin{cases} x^2+y^2=16, \\ 2x+by=12. \end{cases}$

3. $\begin{cases} x^2+y^2=25, \\ 2x+3y=c. \end{cases}$

4. $\begin{cases} y^2=8x, \\ 3x+4y=c. \end{cases}$

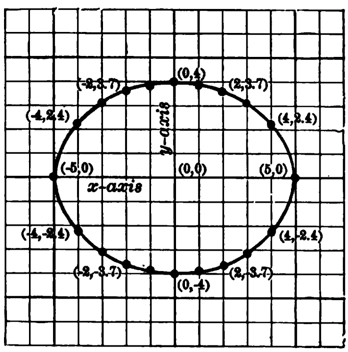
262. Problem. The Equation of the Ellipse. Graph the equation $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1.$

Writing the equation in the form $y=\pm\frac{4}{5}\sqrt{25-x^2}$, and assigning values to x, we compute the corresponding values of y as follows:

$\begin{cases} x=0, \\ y=\pm 4, \end{cases}$ $\begin{cases} x=\pm 5, \\ y=0, \end{cases}$ $\begin{cases} x=1, \\ y=\pm 3.9, \end{cases}$

$\begin{cases} x=-1, \\ y=\pm 3.9, \end{cases}$ $\begin{cases} x=2, \\ y=\pm 3.7, \end{cases}$

$\begin{cases} x=3, \\ y=\pm 3.2, \end{cases}$ $\begin{cases} x=4, \\ y=\pm 2.4. \end{cases}$



Evidently if x is greater than 5 in absolute value, the corresponding values of y are imaginary.

Plotting these points, they are found to lie on the curve shown in the figure.

This curve is called an **ellipse**.

EXERCISES

Solve the following pairs of equations:

From the solutions determine whether the straight line and the curve intersect, and in case they do, find the coördinates of the intersection points. Verify each by constructing the graphs.

1. $\begin{cases} \dfrac{x^2}{16}+\dfrac{y^2}{9}=1, \\ 3x+4y=12. \end{cases}$

2. $\begin{cases} \dfrac{x^2}{49}+\dfrac{y^2}{16}=1, \\ 2x-7y=8. \end{cases}$

3. $\begin{cases} x^2+4y^2=25, \\ 2x-y=4. \end{cases}$

4. $\begin{cases} 3x^2+2y^2=11, \\ x-3y=7. \end{cases}$

5. $\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{9}=1, \\ 2x-y=14. \end{cases}$

6. $\begin{cases} y=2x^2-3x+4, \\ y-4x-8=0. \end{cases}$

When arbitrary constants are introduced in the equations of a straight line and an ellipse, we may determine values for these constants so as to make the line cut the ellipse, touch it, or not cut it, as in the case of the circle, §§ 260, 261.

EXERCISES

Solve each of the following pairs simultaneously:

In each example give such values to the parameter that the line shall (*a*) cut the curve in two distinct points, (*b*) be a tangent to the curve, (*c*) have no point in common with the curve.

1. $\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{16}=1, \\ 4x+y=2. \end{cases}$

2. $\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{b^2}=1, \\ x+5y=5. \end{cases}$

3. $\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{16}=1, \\ 4x-5y=c. \end{cases}$

4. $\begin{cases} \dfrac{x^2}{16}+\dfrac{y^2}{25}=1, \\ 5x-by=20. \end{cases}$

5. $\begin{cases} \dfrac{x^2}{16}+\dfrac{y^2}{25}=1, \\ ax+4y=20. \end{cases}$

6. $\begin{cases} \dfrac{x^2}{36}+\dfrac{y^2}{16}=1, \\ ax+6y-24=0. \end{cases}$

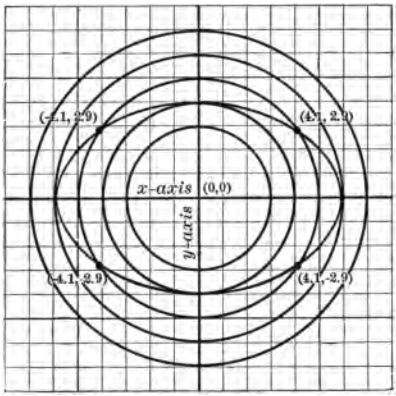
ORAL EXERCISES

In the figure on this page the ellipse is the graph of the equation $\frac{x^2}{36}+\frac{y^2}{16}=1$.

1. What are the radii of the circles? What are their equations?

2. In how many points does each circle meet the ellipse?

3. Is it possible to draw a circle which will meet the ellipse in more than four points?



Example. Solve simultaneously, obtaining results to one decimal place:

$$\begin{cases} \frac{x^2}{36}+\frac{y^2}{16}=1, & (1) \\ x^2+y^2=25. & (2) \end{cases}$$

Clear (1) of fractions and proceed as in § 246. Verify the solution by reference to the graph given in the figure.

EXERCISES

Solve simultaneously each of the following pairs of equations and interpret all the solutions in each case from the graph in the above figure:

1. $\begin{cases} \frac{x^2}{36}+\frac{y^2}{16}=1, \\ x^2+y^2=36. \end{cases}$

2. $\begin{cases} \frac{x^2}{36}+\frac{y^2}{16}=1, \\ x^2+y^2=16. \end{cases}$

3. $\begin{cases} \frac{x^2}{36}+\frac{y^2}{16}=1, \\ x^2+y^2=49. \end{cases}$

4. $\begin{cases} \frac{x^2}{36}+\frac{y^2}{16}=1, \\ x^2+y^2=31. \end{cases}$

263. Problem. Intersections of a Circle and a Parabola.

Solve simultaneously $\begin{cases} x^2 + y^2 = 25, & (1) \\ x^2 = 4y + a. & (2) \end{cases}$

Substituting x^2 from (2) in (1),

$$y^2 + 4y + a - 25 = 0. \qquad (3)$$

Hence, $$y = \frac{-4 \pm \sqrt{16 - 4(a - 25)}}{2} = -2 \pm \sqrt{29 - a}. \qquad (4)$$

Substituting in (2), we have

$$x^2 = -8 \pm 4\sqrt{29 - a} + a$$

and $$x = \pm 2\sqrt{-2 \pm \sqrt{29 - a} + a}.$$

Arranging the values in pairs, we have

$$(1) \quad \begin{cases} x = \sqrt{-8 + 4\sqrt{29 - a} + a}, \\ y = -2 + \sqrt{29 - a}. \end{cases}$$

$$(2) \quad \begin{cases} x = \sqrt{-8 - 4\sqrt{29 - a} + a}, \\ y = -2 - \sqrt{29 - a}. \end{cases}$$

$$(3) \quad \begin{cases} x = -\sqrt{-8 + 4\sqrt{29 - a} + a}, \\ y = -2 + \sqrt{29 - a}. \end{cases}$$

$$(4) \quad \begin{cases} x = -\sqrt{-8 - 4\sqrt{29 - a} + a}, \\ y = -2 - \sqrt{29 - a}. \end{cases}$$

If $a > 29$, all four solutions are imaginary.

If $a = 29$, (1) and (2) are coincident, and (3) and (4) are also coincident.

If $a = 20$, the solutions are:

$$(1) \quad \begin{cases} x = \sqrt{24}, \\ y = 1. \end{cases} \qquad (2) \quad \begin{cases} x = 0, \\ y = -5. \end{cases}$$

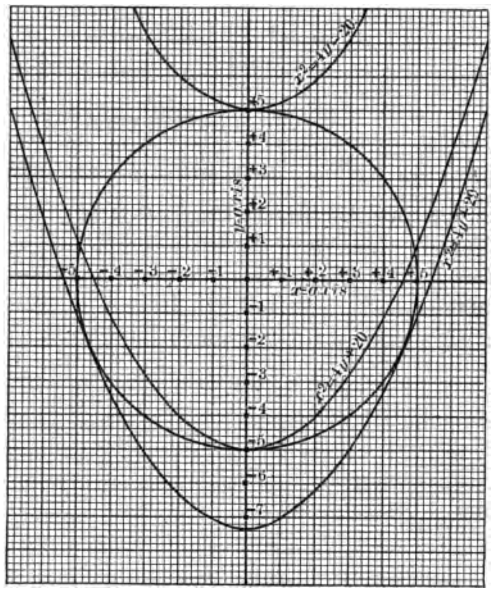
$$(3) \quad \begin{cases} x = -\sqrt{24}, \\ y = 1. \end{cases} \qquad (4) \quad \begin{cases} x = 0, \\ y = -5. \end{cases}$$

That is, (2) and (4) coincide, while (1) and (3) are real and distinct.

If $a = -20$, the solutions are:

$$(1) \quad \begin{cases} x = 0, \\ y = 5. \end{cases} \qquad (2) \quad \begin{cases} x = \sqrt{-56}, \\ y = -9. \end{cases}$$

$$(3) \quad \begin{cases} x = 0, \\ y = 5. \end{cases} \qquad (4) \quad \begin{cases} x = -\sqrt{-56}, \\ y = -9. \end{cases}$$



That is, (1) and (3) are coincident, while (2) and (4) are *imaginary* and *distinct.*

For all values of a between 29 and 20, the solutions are all *real* and *distinct.*

For all values of a between 20 and -20, two solutions are *imaginary* and two are *real* and *distinct.*

For all values of a less than -20, all solutions are *imaginary.*

That is, from the equation $x^2 = 4y + a$, the lowest parabola corresponds to $a = 29$. As a decreases from 29 to 20 the parabola moves up from the lowest position to the second position. As a decreases from 20 to -20, the parabola moves up from the middle position to the upper position, as shown in the figure.

A study of the above graph verifies the correctness of these statements.

WRITTEN EXERCISES

In each of the following, find the values of the parameters for which, (a) the roots are coincident, (b) the roots are real and distinct, (c) the roots are imaginary. (See § 254.)

1. $x^2 + 12x + a = 0.$
2. $x^2 + 4ax + 8 = 0.$
3. $x^2 + ax + a = 0.$
4. $x^2 + 3ax + 2a = 0.$
5. $x^2 + 4x + 2a^2 = 0.$
6. $x^2 + 3ax + a = 0.$
7. $x^2 + 10x - 4a^2 = 0.$
8. $x^2 - 12ax + 20a = 0.$

CHAPTER XVII

PROGRESSIONS

ARITHMETIC PROGRESSIONS

264. An Arithmetic Progression is a series of numbers, such that any one after the first is obtained by adding a fixed number to the preceding. The fixed number is called the **common difference.**

The general form of an arithmetic progression is

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,$$

where a is the first term and d the common difference.

E.g. 2, 5, 8, 11, 14, $\cdots$ is an arithmetic progression in which 2 is the first term and 3 the common difference. Written in the general form, this progression would be

$$2,\ 2+3,\ 2+2\cdot 3,\ 2+3\cdot 3,\ 2+4\cdot 3,\ \cdots.$$

265. The Last Term. From the general form of the arithmetic progression we see that the third term equals the first plus twice the difference, the fourth term equals the first plus three times the difference, and in general if there are n terms in the progression, then the last term is $a+(n-1)d$. Indicating the last term by l, we have

$$l = a + (n-1)d. \qquad \text{I}$$

An arithmetic progression of n terms would then be written in general form thus,

$$a,\ a+d,\ a+2d,\ \cdots,\ a+(n-2)d,\ a+(n-1)d.$$

ORAL EXERCISES

1. If $a = 2$ and $d = 2$, find the 4th term.
2. If $a = 3$ and $d = 4$, find the 5th term.
3. If $a = 3$ and $d = -6$, find the 6th term.

4. If $a = 1$ and $d = \frac{1}{2}$, find the 10th term.
5. If $a = 2$ and $d = -1$, find the 15th term.
6. If $a = -1$ and $d = 3$, find the 12th term.

WRITTEN EXERCISES

1. Solve I for each letter in terms of all the others.

In each of the following find the value of the letter not given, and write out the progression in each case:

2. $\begin{cases} a = 2, \\ d = 2, \\ n = 7. \end{cases}$ 5. $\begin{cases} a = 7, \\ n = 31, \\ l = 91. \end{cases}$ 8. $\begin{cases} a = 3, \\ d = -5, \\ l = -32. \end{cases}$ 11. $\begin{cases} a = -3, \\ n = 9, \\ l = -27. \end{cases}$

3. $\begin{cases} a = 3, \\ d = 5, \\ l = 43. \end{cases}$ 6. $\begin{cases} a = 4, \\ d = -3, \\ n = 18. \end{cases}$ 9. $\begin{cases} d = 7, \\ n = 8, \\ l = 24. \end{cases}$ 12. $\begin{cases} a = 11, \\ l = -39, \\ d = -5. \end{cases}$

4. $\begin{cases} a = 1, \\ n = 15, \\ l = 15. \end{cases}$ 7. $\begin{cases} a = -5, \\ d = 4, \\ n = 7. \end{cases}$ 10. $\begin{cases} d = -5, \\ n = 13, \\ l = -63. \end{cases}$ 13. $\begin{cases} a = x, \\ l = y, \\ n = z. \end{cases}$

266. The Sum of an Arithmetic Progression of n terms may be obtained as follows:

Let s_n denote the sum of n terms. Call the last term l, then the next preceding is $l - d$, the next $l - 2d$, etc. Hence we may write

$$s_n = a + [a + d] + [a + 2d] + \cdots [l - 2d] + [l - d] + l. \quad (1)$$

This may also be written, reversing the order of the terms, thus,

$$s_n = l + [l - d] + [l - 2d] + \cdots + [a + 2d] + [a + d] + a. \quad (2)$$

Adding (1) and (2), we have $(a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l)$, since at each step the d's cancel.

Since there are n of these expressions, each equal to $a + l$, we have

$$2s_n = n(a + l).$$

Hence,
$$s_n = \frac{n}{2}(a + l). \quad \text{II}$$

This formula for the sum of n terms involves a, l, and n, that is, the first term, the last term, and the number of terms.

267. Fundamental Formulas. In the two equations,

$$l = a + (n-1)d, \qquad \text{I}$$

$$s = \frac{n}{2}(a + l), \qquad \text{II}$$

there are five letters, namely, a, d, l, n, s. If any three of these are given, the equations I and II may be solved simultaneously to find the other two.

Solution of Problems.

Example 1. Given $n = 11$, $l = 23$, $s = 143$. Find a and d.

Substituting the given values in I and II,

$$\begin{cases} 23 = a + (11 - 1)d. & (1) \\ 143 = \frac{11}{2}(a + 23). & (2) \end{cases}$$

From (2), $a = 3$, which in (1) gives $d = 2$.

Example 2. Given $d = 4$, $n = 5$, $s = 75$. Find a and l.

From I and II,
$$\begin{cases} l = a + (5 - 1)4, & (1) \\ 75 = \frac{5}{2}(a + l). & (2) \end{cases}$$

Solving (1) and (2) simultaneously, we have $a = 7$, $l = 23$.

Example 3. Given $d = 4$, $l = 35$, $s = 161$. Find a and n.

From I and II,
$$\begin{cases} 35 = a + (n - 1)4, & (1) \\ 161 = \frac{n}{2}(a + 35). & (2) \end{cases}$$

Solving (1) and (2) simultaneously, we get

$$\begin{cases} n = \frac{23}{2}, \\ a = -7. \end{cases} \text{ or } \begin{cases} n = 7, \\ a = 11. \end{cases}$$

Since an arithmetic progression must have an *integral* number of terms, only the second values of n and a are applicable to this problem.

Example 4. Given $d = 2$, $l = 11$, $s = 35$. Find a and n.

Substituting in I and II, and solving for a and n, we have

$$a = 3,\ n = 5, \text{ and } a = -1,\ n = 7.$$

Hence, there are two progressions,

$$-1,\ 1,\ 3,\ 5,\ 7,\ 9,\ 11,$$

and

$$3,\ 5,\ 7,\ 9,\ 11,$$

each of which satisfies the given conditions.

WRITTEN EXERCISES

In each of the following obtain the values of the two letters not given:

Fractional, negative, or imaginary values of n do not satisfy the conditions of the problems. In each exercise interpret all the values found.

1. $\begin{cases} s = 96, \\ l = 19, \\ d = 2. \end{cases}$
2. $\begin{cases} s = 34, \\ l = 14, \\ d = 3. \end{cases}$
3. $\begin{cases} a = 7, \\ l = 27, \\ s = 187. \end{cases}$
4. $\begin{cases} s = 88, \\ l = -7, \\ d = -3. \end{cases}$
5. $\begin{cases} n = 18, \\ a = 4, \\ l = 13. \end{cases}$
6. $\begin{cases} n = 14, \\ a = 7, \\ s = 14. \end{cases}$
7. $\begin{cases} d = -1, \\ n = 41, \\ l = -35. \end{cases}$
8. $\begin{cases} l = 30, \\ s = 162, \\ n = 9. \end{cases}$
9. $\begin{cases} a = 30, \\ n = 10, \\ s = 120. \end{cases}$
10. $\begin{cases} d = 6, \\ l = 49, \\ s = 232. \end{cases}$
11. $\begin{cases} s = 7, \\ d = 1\frac{1}{2}, \\ l = 7. \end{cases}$
12. $\begin{cases} s = 14, \\ d = 3, \\ l = 4. \end{cases}$

In each of the following call the two letters specified the *unknowns* and solve for their values in terms of the remaining three letters in equations I and II.

13. a, d. **14.** a, l. **15.** a, n. **16.** a, s. **17.** d, l. **18.** d, n. **19.** d, s. **20.** l, n. **21.** l, s. **22.** n, s.

268. Arithmetic Means. The terms between the first and the last terms of an arithmetic progression are called **arithmetic means.**

Thus, in 2, 5, 8, 11, 14, 17, the four arithmetic means between 2 and 17 are 5, 8, 11, 14.

If the first and last terms and the number of arithmetic means between them are given, then these means can be found.

For we have given a, l, and n, and hence d can be found and the whole series constructed.

Example. Insert 7 arithmetic means between 3 and 19.

In this case $a = 3$, $l = 19$, and $n = 9$.

Hence, from $l = a + (n-1)d$ we find $d = 2$ and the required means are 5, 7, 9, 11, 13, 15, 17.

269. The case of *one* arithmetic mean is important. Let A be the arithmetic mean between a and l. Since a, A, l are in arithmetic progression, we have $A = a + d$, and $l = A + d$. Hence, eliminating d,

$$l - A = A - a$$

or
$$A = \frac{a + l}{2}. \qquad \text{III}$$

EXERCISES AND PROBLEMS

1. Insert 5 arithmetic means between 5 and -7.

2. Insert 3 arithmetic means between -2 and 12.

3. Insert 8 arithmetic means between -3 and -5.

4. Insert 5 arithmetic means between -11 and 40.

5. Insert 15 arithmetic means between 1 and 2.

6. Insert 9 arithmetic means between $2\frac{3}{4}$ and $-1\frac{1}{2}$.

7. Find the arithmetic mean between 3 and 17.

8. Find the arithmetic mean between -4 and 16.

9. Find the tenth and eighteenth terms of the series 4, 7, 10, $\cdots$.

10. Find the fifteenth and twentieth terms of the series -8, -4, 0, $\cdots$.

11. The fifth term of an arithmetic progression is 13 and the thirtieth term is 49. Find the common difference.

12. Find the sum of all the integers from 1 to 100.

13. Find the sum of all the odd integers between 0 and 200.

14. Find the sum of all integers divisible by 6 between 1 and 500.

15. There are three numbers in arithmetic progression whose sum is 15. The product of the first and last is $3\frac{1}{5}$ times the second. Find the numbers.

16. In a potato race 40 potatoes are placed in a straight line one yard apart, the first potato being two yards from the basket. How far must a contestant travel in bringing them to the basket one at a time?

17. Show that $1 + 3 + 5 + \cdots + n = k^2$, where k is the number of terms.

18. There are four numbers in arithmetic progression whose sum is 20 and the sum of whose squares is 120. Find the numbers.

19. If a body falls from rest 16.08 feet the first second, 48.24 feet the second second, 80.40 the third, etc., how far will it fall in 10 seconds? 15 seconds? t seconds?

20. According to the law indicated in problem 19 in how many seconds will a body fall 1000 feet? s feet?

21. In what time will a body fall 1000 feet if thrown downward with a velocity of 20 feet per second?

If a body is thrown downward with a velocity of v_0 feet per second, then the distance, s, which it falls in t seconds is $v_0 t$ feet plus the distance it would fall if starting from rest.

That is, $s = v_0 t + \frac{1}{2} g t^2$, where $g = 32.16$.

22. With what velocity must a body be thrown downward in order that it shall fall 360 feet in 3 seconds?

23. A stone is dropped into a well, and the sound of its striking the bottom is heard in 3 seconds. How deep is the well if sound travels 1080 feet per second?

24. A rifle bullet is shot directly upward with a velocity of 2000 feet per second. How high will it rise and how long before it will reach the ground?

A body thrown upward with a certain velocity will rise as far as it would have to fall to acquire this velocity. The velocity (neglecting the resistance of the atmosphere) of a body starting from rest is $v = gt$, where $g = 32.16$ and t is the number of seconds.

25. From a balloon 5800 feet above the earth, a body is thrown downward with a velocity of 40 feet per second. In how many seconds will it reach the ground?

26. If in problem 25 the body is thrown upward at the rate of 40 feet per second, how long before it will reach the ground?

GEOMETRIC PROGRESSIONS

270. A Geometric Progression is a series of numbers in which any term after the first is obtained by multiplying the preceding term by a fixed number, called the **common ratio.**

The general form of a geometric progression is

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1},$$

in which a is the first term, r the constant multiplier, or common ratio, and n the number of terms.

E.g. 3, 6, 12, 24, 48, is a geometric progression in which 3 is the first term, 2 is the common ratio, and 5 is the number of terms.

271. The Last Term. If l is the last or nth term of the series, then

$$l = ar^{n-1}. \qquad \text{I}$$

If any three of the four letters in I are given, the remaining one may be found by solving this equation.

WRITTEN EXERCISES

In each of the following find the value of the letter not given:

1. $\begin{cases} l = 162, \\ r = 3, \\ n = 5. \end{cases}$ **4.** $\begin{cases} a = -1, \\ r = -2, \\ n = 9. \end{cases}$ **7.** $\begin{cases} a = -\frac{1}{2}, \\ r = \frac{3}{2}, \\ n = 6. \end{cases}$ **10.** $\begin{cases} l = 32, \\ r = -2, \\ n = 6. \end{cases}$

2. $\begin{cases} a = 1, \\ r = 2, \\ n = 8. \end{cases}$ **5.** $\begin{cases} l = 1024, \\ r = -2, \\ n = 11. \end{cases}$ **8.** $\begin{cases} l = 18, \\ r = \frac{1}{3}, \\ n = 6. \end{cases}$ **11.** $\begin{cases} a = -2, \\ r = -\frac{3}{2}, \\ n = 7. \end{cases}$

3. $\begin{cases} a = -4, \\ r = -3, \\ n = 6. \end{cases}$ **6.** $\begin{cases} l = 1024, \\ r = 2, \\ n = 11. \end{cases}$ **9.** $\begin{cases} l = -16, \\ r = -\frac{3}{4}, \\ n = 5. \end{cases}$ **12.** $\begin{cases} a = 3, \\ r = 2, \\ l = 1536. \end{cases}$

272. The **Sum of n Terms** of a geometric expression may be found as follows:

If s_n denotes the sum of n terms, then

$$s_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \quad (1)$$

Multiplying both members of (1) by r, we have

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \quad (2)$$

Subtracting (1) from (2), and canceling terms, we have

$$rs_n - s_n = ar^n - a. \quad (3)$$

Solving (3) for s_n, we have

$$s_n = \frac{ar^n - a}{r-1} = \frac{a(r^n - 1)}{r-1}. \quad \text{II}$$

This formula for the sum of n terms of a geometric series involves only a, r, and n.

Since $ar^n = r \cdot ar^{n-1} = r \cdot l$, we may also write:

$$s_n = \frac{rl - a}{r-1} = \frac{a - rl}{1-r}. \quad \text{III}$$

This formula involves only s, r, l, and a.

273. Fundamental Formulas. From equations I and II, or I and III, any two of the numbers a, l, r, s, and n can be found when the other three are given, as in the following examples.

Example 1. Given $n = 7$, $r = 2$, $s = 381$. Find a and l.

From I and III, $$l = a \cdot 2^6 = 64\,a, \quad (1)$$

$$381 = \frac{2\,l - a}{2-1} = 2\,l - a. \quad (2)$$

Substituting $l = 64\,a$ in (2), we obtain $a = 3$, and $l = 192$.

Example 2. Given $a = -3$, $l = -243$, $s = -183$. Find r and n.

From I and III, $$-243 = (-3)r^{n-1}, \quad (1)$$

$$-183 = \frac{-243\,r + 3}{r-1}. \quad (2)$$

From (2), $$r = -3. \quad (3)$$

From (1), $$81 = (-3)^{n-1}. \quad (4)$$

Since $(-3)^4 = 81$, we have $n - 1 = 4$ or $n = 5$.

WRITTEN EXERCISES

1. Solve II for a in terms of the remaining letters.

2. Solve III for each letter in terms of the remaining letters.

In each of the following find the terms represented by the interrogation points:

3. $\begin{cases} a=1, \\ r=3, \\ n=5, \\ s=? \end{cases}$ **4.** $\begin{cases} s=635, \\ r=2, \\ n=7, \\ a=? \end{cases}$ **5.** $\begin{cases} s=13, \\ r=\frac{2}{3}, \\ n=4, \\ a=? \end{cases}$ **6.** $\begin{cases} l=-\frac{16}{81}, \\ s=? \\ n=5, \\ r=\frac{2}{3}. \end{cases}$

7. $\begin{cases} a=1, \\ s=\frac{25}{64}, \\ l=-\frac{27}{64}, \\ r=? \end{cases}$ **8.** $\begin{cases} r=6, \\ n=5, \\ l=1296, \\ a=? \\ s=? \end{cases}$ **9.** $\begin{cases} r=\frac{3}{2}, \\ n=8, \\ s=1050\frac{5}{6}, \\ l=? \\ a=? \end{cases}$ **10.** $\begin{cases} a=\frac{9}{2}, \\ n=7, \\ l=\frac{32}{81}, \\ r=? \\ s=? \end{cases}$

274. Geometric Means. The terms between the first term and the last term of a geometric progression are called **geometric means.**

Thus in 3, 6, 12, 24, 48, the terms 6, 12, 24 are geometric means between 3 and 48.

If the first term, the last term, and the number of geometric means are given, the ratio may be found from I, and then the means may be inserted.

Example. Insert 4 geometric means between 2 and 64.

We have given $a=2$, $l=64$, $n=4+2=6$, to find r.

From I, $64 = 2 \cdot r^{6-1}$, or $r^5 = 32$, and $r = 2$.

Hence, the required geometric means are, 4, 8, 16, 32.

275. The case of *one* geometric mean is important. If G is the geometric mean between a and b, we have $\frac{G}{a} = \frac{b}{G}$.

Hence, $$G = \sqrt{ab}.$$

That is, G is a mean proportional between a and b.

EXERCISES AND PROBLEMS

1. Insert 5 geometric means between 2 and 128.
2. Insert 7 geometric means between 1 and $\frac{1}{256}$.
3. Find the geometric mean between 8 and 18.
4. Find the geometric mean between $\frac{1}{12}$ and $\frac{1}{4}$.

Given	Find	Given	Find
5. a, r, n	l, s	9. a, n, l	s, r
6. a, r, s	l	10. r, n, l	s, a
7. r, n, s	l, a	11. r, l, s	a
8. a, r, l	s	12. a, l, s	r

13. The product of three terms of a geometric progression is 1000. Find the second term.

14. Four numbers are in geometric progression. The sum of the second and third is 18, and the sum of the first and fourth is 27. Find the numbers.

15. Find an arithmetic progression whose first term is 1 and whose first, second, fifth, and fourteenth terms are in geometric progression.

16. Three numbers whose sum is 27 are in arithmetic progression. If 1 is added to the first, 3 to the second, and 11 to the third, the sums will be in geometric progression. Find the numbers.

17. To find the amount at compound interest when the principal, the rate of interest, and the time are given.

Solution. Let p equal the number of dollars invested, r the rate per cent of interest, t the number of years, and a the amount at the end of t years.

Then $a = p(1 + r)$ at the end of one year,

$a = p(1 + r)(1 + r) = p(1 + r)^2$ at the end of two years,

and $a = p(1 + r)^t$ at the end of t years. See page 248.

18. Show how to modify the solution given under problem 17 when the interest is compounded semi-annually; quarterly.

276. Infinite Geometric Series. In attempting to reduce $\frac{2}{3}$ to a decimal, we find by division .666 ⋯, the dots indicating that the 6's are repeated indefinitely.

Conversely, we see that .666 ... $= \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots$; that is, a geometric progression in which $a = \frac{6}{10}$, $r = \frac{1}{10}$, and n is not fixed but *goes on increasing indefinitely.*

As n grows large, l grows small, and by taking n sufficiently large, *l can be made as small as we please.*

Hence, formula III, § 272, is to be interpreted in this case as follows:

$$s_n = \frac{a - rl}{1 - r} = \frac{\frac{6}{10} - \frac{1}{10} \cdot l}{1 - \frac{1}{10}} = \frac{6 - l}{9},$$

in which *l grows small indefinitely as n increases indefinitely*, so that, by taking n large enough, s_n can be made to *differ as little as we please* from $\frac{6-0}{9} = \frac{6}{9} = \frac{2}{3}$.

In this case we say *s_n approaches $\frac{2}{3}$ as a limit as n increases indefinitely.*

Similarly, if we have .3333 ⋯ or $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$, then s_n approaches $\frac{1}{3}$ as a limit.

277. Sum of an Infinite Geometric Series. The series $a + ar + ar^2 + \cdots + ar^n + \cdots$, where the number of terms goes on indefinitely, is called an **infinite geometric series.**

The limit of the sum of the *general infinite geometric series* may be found in the above manner, *provided r is a proper fraction.*

The sum s_n of the first n terms is

$$s_n = \frac{a - rl}{1 - r} = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

If r is a *proper fraction*, then *r^n approaches nearer and nearer to zero as n becomes large*, and hence $\frac{ar^n}{1 - r}$ *becomes as small as we please.*

Hence, $\dfrac{a - ar^n}{1 - r}$ can be made to approach as nearly as we please to $\dfrac{a}{1 - r}$, by taking n large enough.

Hence, s_n approaches $\dfrac{a}{1 - r}$ as n increases indefinitely.

Hence, $\dfrac{a}{1 - r}$ is said to be *the sum of the infinite series,*

$$a + ar + ar^2 + \cdots + ar^n + \cdots,$$

provided r lies between -1 and $+1$.

If r does not lie between -1 and $+1$, then r^n does not approach zero, and hence the sum of n terms of the infinite geometric series does not approach any limit as n increases indefinitely.

Example. Find the fraction which is the limit of .88888 ....

Solution. $a = \frac{8}{10}$, $r = \frac{1}{10}$.

Hence, $s_n = \dfrac{a - ar^n}{1 - r} = \dfrac{\frac{8}{10} - \frac{8}{10}\cdot(\frac{1}{10})^n}{1 - \frac{1}{10}} = \dfrac{\frac{8}{10}}{\frac{9}{10}} - \dfrac{\frac{8}{10}(\frac{1}{10})^n}{\frac{9}{10}} = \dfrac{8}{9} - \dfrac{8}{9}\cdot\left(\dfrac{1}{10}\right)^n.$

The limit as n increases indefinitely is therefore $\frac{8}{9}$, which is the required fraction.

WRITTEN EXERCISES

1. Find the fraction which is the limit of .333 ....

2. Find the fraction which is the limit of .1666 ....

Suggestion. The series is $\frac{1}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots$. Find the sum of the series beginning with $\frac{6}{100}$ and then add $\frac{1}{10}$.

3. Find the fraction which is the limit of .08333 ....

4. Find the fractions which are the limits of .1444 ..., .07555, and .19222.

5. Find the fraction which is the limit of .909090.

Suggestion. The series is $\frac{9}{10} + \frac{9}{1000} + \frac{9}{100000} + \cdots$.

6. Find the fraction which is the limit of .141414 ....

7. Find the sum of the infinite series $-\frac{16}{3} + 4 - 3 + \frac{9}{4} \cdots$.

8. Find the sum of the series $-243 + 81 - 27 + \cdots$.

9. Find the sum $\frac{4}{3} + \frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \cdots$.

HARMONIC PROGRESSIONS

278. A **Harmonic Progression** is a series whose terms are the reciprocals of the corresponding terms of an arithmetic progression.

E.g. $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9} \cdots$ is a harmonic progression whose terms are the reciprocals of the terms of the arithmetic progression $1, 3, 5, 7, 9 \cdots$.

The name *harmonic* is given to such a series because musical strings of uniform size and tension vibrate in harmony if their lengths are proportional to the reciprocals of the positive integers, *i.e.* to $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots$.

279. The general form of the harmonic progression is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots, \frac{1}{a+(n-1)d}. \qquad \text{I}$$

It follows that if $a, b, c, d, e, \cdots$ are in harmonic progression, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}, \cdots$ are in arithmetic progression. Hence, all questions pertaining to a harmonic progression may be answered by first converting it into an *arithmetic progression.*

280. Harmonic Means. The terms between the first and the last terms of a harmonic progression are called **harmonic means** between them.

Example. Insert five harmonic means between 30 and 3.

This is done by inserting five arithmetic means between $\frac{1}{30}$ and $\frac{1}{3}$. By the method of § 268 the arithmetic series is found to be $\frac{1}{30}, \frac{1}{12}, \frac{2}{15}, \frac{11}{60}, \frac{7}{30}, \frac{17}{60}, \frac{1}{3}$. Hence, the harmonic series is $30, 12, \frac{15}{2}, \frac{60}{11}, \frac{30}{7}, \frac{60}{17}, 3$.

281. The case of a single harmonic mean is important. Let a, H, l be in harmonic progression. Then $\frac{1}{a}, \frac{1}{H}, \frac{1}{l}$ are in arithmetic progression.

Hence, by § 269, $\dfrac{1}{H} = \dfrac{\frac{1}{a}+\frac{1}{l}}{2}$ or $H = \dfrac{2al}{a+l}$.

282. The *arithmetic, geometric,* and *harmonic means* between a and l are related as follows:

We have seen $A=\frac{a+l}{2}$, $G=\sqrt{al}$, $H=\frac{2\,al}{a+l}$.

Hence, $\frac{A}{G^2}=\frac{a+l}{2}\div al=\frac{a+l}{2\,al}$.

Therefore, $\frac{A}{G^2}=\frac{1}{H}$, or $\frac{A}{G}=\frac{G}{H}$.

That is, G is a geometric mean between A and H. See § 275.

283. If a and l are both positive, then A and H are positive, and therefore G is a real number and G lies between A and H.

Moreover, by reducing $\frac{a+l}{2}$ and $\frac{2\,al}{a+l}$ to a common denominator we get $A=\frac{(a+l)^2}{2(a+l)}$ and $H=\frac{4\,al}{2(a+l)}$.

Hence we see that $A>H$, since $a^2+l^2+2\,al-4\,al=a^2-2\,al+l^2=(a-l)^2>0$.

Hence, of the three means the arithmetic mean is the greatest and the harmonic is least.

EXERCISES AND PROBLEMS

1. Insert three harmonic means between 22 and 11.

2. Insert six harmonic means between $\frac{1}{3}$ and $\frac{23}{6}$.

3. The first term of a harmonic progression is $\frac{1}{2}$ and the tenth term is $\frac{1}{20}$. Find the intervening terms.

4. Two consecutive terms of a harmonic progression are 5 and 6. Find the next two terms and also the two preceding terms.

5. If a, b, c are in harmonic progression, show that $a\div c=(a-b)\div(b-c)$.

6. Find the arithmetic, geometric, and harmonic means between:

(*a*) 16 and 36; (*b*) $m+n$ and $m-n$; (*c*) $\dfrac{1}{m+n}$ and $\dfrac{1}{m-n}$.

7. The harmonic mean between two numbers exceeds their arithmetic mean by 7, and one number is three times the other. Find the numbers.

8. If x, y, and z are in arithmetic progression, show that mx, my, and mz are also in arithmetic progression.

9. x, y, and z being in harmonic progression, show that $\dfrac{x}{x+y+z}$, $\dfrac{y}{x+y+z}$, and $\dfrac{z}{x+y+z}$ are in harmonic progression, and also that $\dfrac{x}{y+z}$, $\dfrac{y}{x+z}$, and $\dfrac{z}{x+y}$ are in harmonic progression.

10. The sum of three numbers in harmonic progression is 3, and the third is double the first. Find the numbers.

11. The geometric mean between two numbers is $\frac{1}{4}$ and the harmonic mean is $\frac{1}{5}$. Find the numbers.

12. Insert n harmonic means between the numbers a and b.

13. The first term of a harmonic progression is a and the second is b. Write the 3d and the 4th terms.

14. The arithmetic mean of two numbers is 2 and the harmonic mean is $\frac{3}{2}$. Find the numbers.

15. The first term of a harmonic progression is $\frac{3}{4}$ and the 7th term is $6\frac{3}{4}$. Find the progression.

16. Find two numbers such that their geometric mean is 4 and their harmonic mean $\frac{5}{16}$.

17. The geometric mean of two numbers is 6 and their arithmetic mean is $7\frac{1}{2}$. Find the numbers.

18. a^2, b^2, c^2, $\cdots$ are in arithmetic progression. Prove that $b+c$, $c+a$, $a+b$ are in harmonic progression.

CHAPTER XVIII

THE BINOMIAL FORMULA

284. By actual multiplication the following products are obtained:

$$(a+b)^2 = a^2 + 2\,ab + b^2.$$
$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$
$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$$
$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5.$$

In these products the following rule is seen to hold:

1. The number of terms in each product is one greater than the exponent of the binomial.

2. The exponent of a in the *first* term is the same as the exponent of the binomial, and diminishes by unity in each *succeeding* term.
The exponent of b in the *last* term is the same as the exponent of the binomial, and diminishes by unity in each *preceding* term.

3. The sum of the exponents in each term is equal to the exponent of the binomial.

4. The coefficient of the first term is unity; of the second term, the same as the exponent of the binomial; and the coefficient of any other term may be found by multiplying the coefficient of the next preceding term by the exponent of a in that term and dividing this product by a number one greater than the exponent of b in that term.

5. The coefficients of any pair of terms equally distant from the ends are equal.

Let us now find $(a+b)^6$.

$$(a+b)^6 = (a+b)\,(a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,b^4 + b^5)$$
$$= \begin{cases} a^6 + 5\,a^5b + 10\,a^4b^2 + 10\,a^3b^3 + 5\,a^2b^4 + ab^5 \\ \qquad\quad a^5b + 5\,a^4b^2 + 10\,a^3b^3 + 10\,a^2b^4 + 5\,ab^5 + b^6 \end{cases}$$

Hence $(a+b)^6 = a^6 + 6\,a^5b + 15\,a^4b^2 + 20\,a^3b^3 + 15\,a^2b^4 + 6\,ab^5 + b^6$

From this it is seen that the rule holds also for $(a+b)^6$.

PROOF BY MATHEMATICAL INDUCTION

285. A proof that the above rule holds for *all positive integral powers* of a binomial may be made as follows:

First step. Write out the product as it *would be* for the nth power on the *supposition that the rule holds.*

Then the first term would be a^n and the last term b^n. The second terms from the ends would be $na^{n-1}b$ and nab^{n-1}. The third terms fron the ends would be $\frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2$ and $\frac{n(n-1)}{1 \cdot 2}a^2b^{n-2}$. The fourth terms from the ends would be

$$\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 \text{ and } \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^3b^{n-3},$$

and so on, giving

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \cdots + \frac{n(n-1)}{1 \cdot 2}a^2b^{n-2} + nab^{n-1} + b^n.$$

Second step. Multiply this expression by $a + b$ and see if the result, namely, $(a + b)^{n+1}$, can be so arranged as to conform to the same rule.

Multiplying the above expression for $(a + b)^n$ by a and then by b, we have

$$\left\{\begin{array}{l} a^{n+1} + na^nb + \frac{n(n-1)}{1 \cdot 2}a^{n-1}b^2 + \cdots \qquad + na^2b^{n-1} + ab^n \\ \qquad a^nb \qquad + na^{n-1}b^2 + \cdots + \frac{n(n-1)}{1 \cdot 2}a^2b^{n-1} + nab^n + b^{n+1} \end{array}\right.$$

Hence, adding,

$$(a + b)^{n+1} = a^{n+1} + (n + 1)a^nb + \left[\frac{n(n-1)}{1 \cdot 2} + n\right]a^{n-1}b^2 + \cdots$$
$$+ \left[n + \frac{n(n-1)}{1 \cdot 2}\right]a^2b^{n-1} + (n + 1)ab^n + b^{n+1}.$$

Combining the terms in brackets, we have,

$$(a + b)^{n+1} = a^{n+1} + (n + 1)a^nb + \frac{(n+1)n}{1 \cdot 2}a^{n-1}b^2 + \cdots$$
$$+ \frac{(n+1)n}{1 \cdot 2}a^2b^{n-1} + (n + 1)ab^n + b^{n+1}.$$

The last result shows that the rule holds for $(a + b)^{n+1}$ *if it holds for* $(a + b)^n$. That is, *if the rule holds for any positive integral exponent, it holds for the next higher integral exponent.*

Third step. It was found above by *actual multiplication* that the rule does hold up to $(a + b^6)$. Hence by the above argument we know that the rule holds for $(a + b)^7$.

Moreover, since we now know that the rule holds for $(a + b)^7$, we conclude by the same argument that it holds for $(a + b)^8$, and if for $(a + b)^8$, then for $(a + b)$,^{9} and so on.

Since this process of extending to higher powers can be carried on indefinitely, we conclude that the five statements in § 284 *hold for all positive integral powers of a binomial.*

The essential step in this proof consists in showing that the rule does hold for the $(n + 1)$th power *if it holds for the nth power.*

286. The General Term. According to the rule now known to hold for any positive integral exponent, we may write as many terms of the expansion of $(a + b)^n$ as may be desired, thus:

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2$$
$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}a^{n-4}b^4 + \cdots \quad \text{I}$$

From this result, called the **binomial formula**, we see:

(1) *The exponent of b in any term is one less than the number of that term, and the exponent of a is n minus the exponent of b. Hence, the exponent of b in the $(k+1)$st term is k, and the exponent of a is $n - k$.*

(2) *In the coefficient of any term the last factor in the denominator is the same as the exponent of b in that term, and the last factor in the numerator is one greater than the exponent of a.*

Hence the $(k + 1)$st term, which is called the **general term**, is

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots k}a^{n-k}b^k. \quad \text{II}$$

E.g. In $(x + y)^{12}$ the 5th term is $\frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4}a^{12-4}b^4$
$= 11 \cdot 5 \cdot 9 \cdot a^8b^4.$

287. Expansion of a Binomial. The process of writing out a power of a binomial is called **expanding the binomial,** and the result is called an **expansion of the binomial.**

Example 1. Expand $(x-y)^4$.

In this case $a=x,\ b=-y,\ n=4$.

Hence, substituting in formula I, omitting the factor 1,

$$(x-y)^4 = x^4 + 4\,x^3(-y) + \frac{4(4-1)}{2}x^2(-y)^2 + \frac{4(4-1)(4-2)}{2\cdot 3}x(-y)^3$$
$$+\frac{4(4-1)(4-2)(4-3)}{2\cdot 3\cdot 4}(-y)^4 \qquad (1)$$

$$= x^4 - 4\,x^3y + \frac{4\cdot 3}{2}x^2y^2 - \frac{4\cdot 3\cdot 2}{2\cdot 3}xy^3 + \frac{4\cdot 3\cdot 2}{2\cdot 3\cdot 4}y^4. \qquad (2)$$

Hence, $(x-y)^4 = x^4 - 4\,x^3y + 6\,x^2y^2 - 4\,xy^3 + y^4$. (3)

Notice that this is precisely the same as the expansion of $(x+y)^4$ except that every other term beginning with the second is *negative.*

Example 2. Expand $(1-2\,y)^5$.

Here $a=1,\ b=-2\,y,\ n=5$.

Since the coefficients in the expansion of $(a+b)^5$ are 1, 5, 10, 10, 5, 1, we write at once,

$$\begin{aligned}(1-2\,y)^5 &= 1^5 + 5\cdot 1^4\cdot(-2\,y) + 10\cdot 1^3\cdot(-2\,y)^2\\ &\quad + 10\cdot 1^2\cdot(-2\,y)^3 + 5\cdot 1\cdot(-2\,y)^4 + (-2\,y)^5\\ &= 1 - 10\,y + 40\,y^2 - 80\,y^3 + 80\,y^4 - 32\,y^5.\end{aligned}$$

Example 3. Expand $\left(\frac{1}{x}+\frac{y}{3}\right)^5$.

Remembering the coefficients just given, we write at once,

$$\begin{aligned}\left(\frac{1}{x}+\frac{y}{3}\right)^5 &= \left(\frac{1}{x}\right)^5 + 5\left(\frac{1}{x}\right)^4\left(\frac{y}{3}\right) + 10\left(\frac{1}{x}\right)^3\left(\frac{y}{3}\right)^2 + 10\left(\frac{1}{x}\right)^2\left(\frac{y}{3}\right)^3\\ &\quad + 5\left(\frac{1}{x}\right)\left(\frac{y}{3}\right)^4 + \left(\frac{y}{3}\right)^5\\ &= \frac{1}{x^5} + \frac{5\,y}{3\,x^4} + \frac{10\,y^2}{9\,x^3} + \frac{10\,y^3}{27\,x^2} + \frac{5\,y^4}{81\,x} + \frac{y^5}{243}.\end{aligned}$$

In a similar manner any positive integral power of a binomial may be written.

Example 4. Write the *sixth term* in the expansion of $(x-2y)^{10}$ without computing any other term.

From II, § 286, we know the $(k+1)$st term for the nth power of $a+b$, namely,

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{2\cdot 3\cdot 4\cdots k}a^{n-k}b^{k}.$$

In this case $a=x$, $b=-2y$, $n=10$, $k+1=6$. Hence, $k=5$.

Substituting these particular values, we have

$$\begin{aligned}&\frac{10(10-1)(10-2)\cdots(10-5+1)}{2\cdot 3\cdot 4\cdot 5}x^{10-5}(-2y)^5\\ &=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{2\cdot 3\cdot 4\cdot 5}x^5(-2y)^5\\ &=252\,(-32)\,x^5y^5=-8064\,x^5y^5.\end{aligned}$$

WRITTEN EXERCISES

1. Make a list of the coefficients for each power of a binomial from the 2d to the 10th.

Expand the following:

2. $(x-y)^3$.
3. $(2x+3)^3$.
4. $(3x+2y)^4$.
5. $(3+y)^5$.
6. $(x^3+y)^6$.
7. $(x-y^3)^6$.
8. $(x^3-y^2)^7$.
9. $(x^{\frac{1}{2}}-y^{\frac{1}{2}})^4$.
10. $(x^{-1}+y^{-2})^5$.
11. $(a-b)^8$.
12. $(x+y)^9$.
13. $(m-n)^{10}$.
14. $(r^{\frac{1}{2}}+s^2)^4$.
15. $(c^{-2}-d^{-\frac{1}{2}})^5$.
16. $(\sqrt{a}-\sqrt{b})^6$.
17. $\left(\frac{x^2}{y}-\frac{y^2}{x}\right)^3$.
18. $\left(\frac{2x}{y^2}-y\sqrt{x}\right)^3$.
19. $\left(\frac{\sqrt{m}}{\sqrt[3]{n^2}}+\sqrt{\frac{y}{n}}\right)^4$.
20. $\left(\frac{c\sqrt[3]{c}}{\sqrt[5]{d^4}}-\frac{\sqrt[3]{d}}{c}\right)^7$.
21. $(2a^2x^{-2}-3by^{-3})^4$.
22. $(3xy^{-3}-x^{-3}y)^8$.
23. $(a^3b^{-1}-a^2b^3)^5$.
24. $(a^{\frac{2}{3}}b^{\frac{3}{2}}+a^{\frac{1}{2}}b^{\frac{1}{3}})^4$.
25. $(x^{-1}y^{\frac{1}{2}}+x^{\frac{1}{2}}y^{-1})^6$.
26. $(3x^{-2}y^2+2xy^{-1})^4$.
27. $(2x^{-1}y^{-3}-x^2y^2)^5$.
28. $(m^3n^{-2}+m^{\frac{1}{2}}n^{\frac{1}{4}})^7$.

In each of the following find the required term without finding any other term:

29. The 5th term of $(a+b)^{12}$.

30. The 7th term of $(3x-2y)^{11}$.

31. The 6th term of $(\sqrt{x}-\sqrt[3]{y})^{10}$.

32. The 9th term of $(x-y)^{25}$.

33. The 8th term of $(\frac{1}{2}m-\frac{1}{8}n)^{18}$.

34. The 7th term of $(a^2b-ab^2)^{30}$.

35. The 6th term of $(a-a^{-1})^{2k}$.

36. The 7th term of $(x^2y-x^{-2}y^{-1})^{3m}$.

37. The 5th term from each end of the expansion of $(a-b)^{20}$.

38. The 7th term from each end of $(a\sqrt{a}-b\sqrt{b})^{21}$.

39. Which term, counting from the beginning, in the expansion of $(a+b)^{10}$ has the same coefficient as the 7th term? Verify by finding both coefficients. How do the exponents differ in these terms?

40. What other term has the same coefficient as the 19th term of $(a+b)^{24}$? How do the exponents differ? Find in the shortest way the 21st term of $(a+b)^{25}$.

41. Find the 87th term of $(a+b)^{90}$.

42. Find the 53d term of $(a^{\frac{1}{2}}-b^{\frac{1}{3}})^{56}$.

43. What other term has the same coefficient as the 5th term in the expansion of $(x+y)^{19}$?

44. Expand $[(a+b)+c]^3$ by the binomial formula.

45. Expand $[1+(2x+3y)]^4$ by the binomial formula.

46. Expand $(2x-3y+4z)^3$ by the binomial formula.

47. Write the $(k+1)$st term of $(a+b)^n$. Write the $(n+1)$st term of $(a+b)^n$. Show that the next and also all succeeding terms after the $(n+1)$st term have zero coefficients, thus proving that there are exactly $n+1$ terms in the expansion.

CHAPTER XIX

LOGARITHMS

288. Logarithms. The operations of multiplication, division, and finding powers and roots are greatly shortened by the use of *logarithms.*

The logarithm of a number, in the system commonly used, is the *index of that power of* 10 *which equals the given number.*

Thus, 2 is the logarithm of 100 since $10^2 = 100$.

This is written $\log 100 = 2.$

Similarly $\log 1000 = 3$, since $10^3 = 1000$,

and $\log 10000 = 4$, since $10^4 = 10000$.

If the logarithm of a positive number is not an integer it may be written approximately or exactly as a decimal fraction.

Thus, $\log 74 = 1.8692$ since $10^{1.8692} = 74$ approximately.

In higher algebra it is shown that the laws for rational exponents (§ 176) hold also for irrational exponents.

289. Mantissa, Characteristic. The decimal part of a logarithm is called the **mantissa,** and the integral part the **characteristic.**

Since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc., it follows that for all numbers between 1 and 10 the logarithm lies between 0 and 1, that is, the characteristic is 0. Likewise for numbers between 10 and 100 the characteristic is 1, for numbers between 100 and 1000 it is 2, etc.

290. Tables of logarithms (see p. 244) give the mantissas only, the characteristics being supplied, according to the rule of § 292.

291. An important property of logarithms is illustrated by the following:

From the table of logarithms, page 244, we have:

$$\log 376 = 2.5752, \text{ or } 376 = 10^{2.5752}. \qquad (1)$$

Dividing both members of (1) by 10, we have

$$37.6 = 10^{2.5752} \div 10^1 = 10^{2.5752-1} = 10^{1.5752}.$$

Hence, $\log 37.6 = 1.5752.$

Similarly, $\log 3.76 = 1.5752 - 1 = 0.5752,$

$\log .376 = 0.5752 - 1, \text{ or } \bar{1}.5752,$

$\log .0376 = 0.5752 - 2, \text{ or } \bar{2}.5752,$

where $\bar{1}$ and $\bar{2}$ are written for -1 and -2 to indicate that the characteristics are negative while the mantissas are positive.

Multiplying (1) by 10 gives

$\log 3760 = 2.5752 + 1 = 3.5752,$

and $\log 37600 = 2.5752 + 2 = 4.5752.$

Hence, if the decimal point of a number is moved a certain number of places to the right or to the left, the characteristic of the logarithm is increased or decreased by a corresponding number of units, the mantissa remaining the same.

From the table on pages 244, 245, we may find the mantissas of logarithms for all integral numbers from 1 to 1000. In this table the logarithms are given to four places of decimals, which is sufficiently accurate for most practical purposes.

Example 1. Find log 876.

Solution. Look down the column headed N to 87, then along this line to the column headed 6, where we find the number 9425, which is the mantissa. Hence, $\log 876 = 2.9425$.

Example 2. Find log 3747.

Solution. As above we find $\log 3740 = 3.5729,$

and $\log 3750 = 3.5740.$

The difference between these logarithms is 11, which corresponds to a difference of 10 between the numbers. But 3740 and 3747 differ by 7. Hence, their logarithms should differ by $\frac{7}{10}$ of 11, *i.e.* by 7.7. Adding 8 (the nearest integer to 7.7) to the logarithm of 3740, we have 3.5737. This is the required logarithm.

Example 3. Find the number whose logarithm is 2.3948.

Solution. Looking in the table of mantissas, we find that the nearest *lower* logarithm is 2.3945, which corresponds to the number 248.

The given mantissa is 3 greater than that of 248, while the mantissa of 249 is 17 greater. Hence, the number corresponding to 2.3948 must be 248 plus $\frac{3}{17}$ of 1, that is, $248 + .18 = 248.18$.

Example 4. Find log .043.

Solution. Find log 43 and subtract 3 from the characteristic.

Example 5. Find the number whose logarithm is $\bar{4}.3949$.

Solution. Find the number whose logarithm is 0.3949, and move the decimal point 4 places to the left.

292. Rule for Finding the Characteristic.

(1) *For numbers greater than 1, the characteristic is positive or zero and always one less than the number of digits to the left of the decimal point.*

(2) *For numbers less than 1, the characteristic is negative and numerically one greater than the number of zeros to the right of the decimal point.*

WRITTEN EXERCISES

Find the logarithms of the following numbers:

1. 491.	**6.** .541.	**11.** .006.	**16.** 79.31.
2. 73.5.	**7.** .051	**12.** .1902.	**17.** 4.245.
3. 2485.	**8.** 8104.	**13.** .0104.	**18.** .0006.
4. 539.7.	**9.** 70349.	**14.** 2.176.	**19.** 3.817.
5. 53.27.	**10.** 439.26.	**15.** 8.094.	**20.** .1341.

Find the numbers corresponding to the following logarithms:

21. 1.3179.	**26.** 2.9900.	**31.** $\bar{1}.5972$.	**36.** 0.2468.
22. 3.0146.	**27.** 0.1731.	**32.** $\bar{1}.0011$.	**37.** 0.1357.
23. 0.7145.	**28.** 0.8974.	**33.** $\bar{2}.7947$.	**38.** $\bar{2}.0246$.
24. $\bar{1}.5983$.	**29.** 0.9171.	**34.** $\bar{2}.5432$.	**39.** $\bar{1}.1358$.
25. 2.0013.	**30.** 3.4015.	**35.** 0.5987.	**40.** $\bar{4}.0478$.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	7857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8318
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

293. Products and powers may be found by means of logarithms, as shown by the following examples.

Example 1. Find the product $49 \times 134 \times .071$.

Solution. From the table,

$$\log 49 = 1.6902 \text{ or } 49 = 10^{1.6902}$$
$$\log 134 = 2.1271 \text{ or } 134 = 10^{2.1271}$$
$$\log .071 = \bar{2}.8513 \text{ or } .071 = 10^{\bar{2}.8513}$$

Since powers of the same base are multiplied by *adding* exponents, we have $49 \times 134 \times .071 = 10^{2.6686}$.

Hence, $\log(49 \times 134 \times .071) = 2.6686$.

The number corresponding to this logarithm, as found by the method used in Example 3, page 243, is 466.2. This is an error of about 1 in 30,000 and is therefore so small as to be negligible.

Example 2. Find $\log (1.05)^{20}$.

Solution. $(1.05)^{20} = (10^{0.0212})^{20} = 10^{(0.0212)20} = 10^{0.424}$.

Hence, $\log(1.05)^{20} = 0.4240$.

It follows from the laws of exponents that

(*a*) *The logarithm of the product of two or more numbers is the sum of the logarithms of the numbers.*

(*b*) *The logarithm of a power of a number is the logarithm of the number multiplied by the index of the power.*

That is,

$$\log(a \cdot b \cdot c) = \log a + \log b + \log c \text{ and } \log a^n = n \log a.$$

WRITTEN EXERCISES

By means of logarithms obtain the following products:

1. $243 \times 76 \times .34$.
2. 823.68×3.70.
3. 216.83×2.03.
4. $(57)^2 \times (.71)^2$.
5. $510 \times (9.1)^3$.
6. $43.71 \times (21)^2$.
7. 5.93×10.02.
8. 486×3.45.
9. $(.02)^2 \times 0.8$.
10. $(6.5)^2 \times (.91)^3$.
11. $(8.4)^2 \times (.75)^3$.
12. $(.96)^2 \times (49)^2$.
13. $(49)^3 \times .19 \times .21^2$.
14. $.21084 \times (.53)^2$.
15. $7.865 \times (.013)^2$.
16. $(6.75)^3 \times (7.23)^2$.
17. $(1.46)^2 \times (61.2)^2$.
18. $(3.54)^3 \times (29.3)^2$.
19. $(4.132)^2 \times (5.184)^2$.
20. $1946 \times 3.98 \times .08$.

294. Quotients and roots may be found by means of logarithms as shown by the following examples.

Example 1. Divide 379 by 793.

Solution. From the table,

$$\log 379 = 2.5786 \text{ or } 10^{2.5786} = 379.$$
$$\log 793 = 2.8993 \text{ or } 10^{2.8993} = 793.$$

Hence, by the law of exponents for division, § 176,

$$379 \div 793 = 10^{2.5786-2.8993}.$$

Since in all operations with logarithms the mantissa is positive, write the first exponent 3.5786 − 1 and then subtract 2.8993.

Hence, $\log (379 \div 793) = .6793 - 1 = \bar{1}.6793.$

Hence, the quotient is the number corresponding to this logarithm.

Example 2. By means of logarithms approximate $\sqrt[3]{42^2 \times 37^5}$.

By the methods used above we find

$$\log (42^2 \times 37^5) = 11.0874 \text{ or } 10^{11.0874} = 42^2 \times 37^5.$$

Hence, $\sqrt[3]{42^2 \times 37^5} = (10^{11.0874})^{\frac{1}{3}} = 10^{\frac{11.0874}{3}} = 10^{3.6958}.$

That is, $\log \sqrt[3]{42^2 \times 37^5} = 3.6958.$

Hence, the result sought is the number corresponding to this logarithm.

It follows from the laws of exponents that

(*a*) *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

(*b*) *The logarithm of a root of a number is the logarithm of the number divided by the index of the root.*

That is,

$$\log \frac{a}{b} = \log a - \log b \text{ and } \log \sqrt[n]{a} = \frac{\log a}{n}.$$

NOTE. — A formula for finding cube roots of numbers is given on pages 126–131. Still higher roots may also be found by the same process, but the work is laborious and complicated. For practical purposes logarithms should be used for finding cube and higher roots, and even for square roots logarithms are desirable.

EXERCISES

By means of logarithms approximate the following quotients and roots:

1. $\sqrt[3]{275}$.
2. $\sqrt[3]{3412}$.
3. $\sqrt[3]{1.72}$.
4. $\sqrt[3]{.00146}$.
5. $\sqrt[3]{347}$.
6. $\sqrt[4]{72.48}$.
7. $\sqrt[5]{11.56}$.
8. $\sqrt[7]{648}$.
9. $\sqrt[6]{54324}$.
10. $\sqrt[8]{.6843}$.
11. $45.2 \div 8.9$.
12. $231.18 \div 4.2$.
13. $.04905 \div .327$.
14. $\sqrt{196 \times 256}$.
15. $\dfrac{5334 \times .02374}{27.43 \times 3.246}$.
16. $\sqrt[5]{69} \div \sqrt[3]{21}$.
17. $\sqrt[7]{15} \times \sqrt[8]{67}$.
18. $\sqrt[10]{211} \times \sqrt[11]{34.7}$.
19. $(5184)^{\frac{1}{2}} \div (38124)^{\frac{1}{3}}$.
20. $(6.75)^3 \div (2.132)^2$.
21. $\sqrt[9]{105} \div \sqrt[13]{76}$.
22. $(91125)^{\frac{1}{3}} \div (576)^{\frac{1}{3}}$.
23. $(3.04)^3 \div (.65)^3$.
24. $(29.3)^{\frac{1}{3}} \div \sqrt{(3.47)^3}$.
25. $\sqrt[3]{39} \times \sqrt[3]{56} \times \sqrt[4]{87}$.
26. $\sqrt[3]{\dfrac{13^4 \times .31^2 \times 4.31^3}{\sqrt{71} \times \sqrt[3]{41} \times \sqrt{51}}}$.
27. $\sqrt[5]{\dfrac{4^9 \times .57^3 \times 42^3}{\sqrt[3]{5.2} \times \sqrt[5]{.83} \times \sqrt{23}}}$.
28. $\left(\dfrac{\sqrt[3]{54} \times \sqrt[4]{28} \times \sqrt[5]{7}}{\sqrt{4^7} \times \sqrt[3]{7^4} \times (.003)^{\frac{3}{4}}}\right)^{\frac{4}{5}}$.

PROBLEMS

1. Solve $a = p(1 + r)^t$ for p and r.

2. Solve $a = p(1 + r^t)$ for t.

Solution. $\log a = \log p\,(1 + r)^t = \log p + \log (1 + r)^t$
$= \log p + t \log (1 + r)$. (See § 293.) Hence $t = \dfrac{\log a - \log p}{\log (1 + r)}$.

3. At what rate of interest compounded annually will \$1200 amount to \$1800 in 12 years?

4. At what rate of interest compounded semi-annually will a sum double itself in 20 years? in 15 years? in 10 years?

5. In what time will \$8000 amount to \$13,500, the rate of interest being $3\frac{1}{2}$ % compounded annually?

6. In what time will a sum double itself at 3 %, 4 %, 5 %, compounded semi-annually?

The present value of a debt due at some future time is a sum such that, if invested at compound interest, the amount at the end of the time will equal the debt.

7. What is the present value of $2500 due in 4 years, money being worth $3\frac{1}{2}$ %, interest compounded semi-annually?

8. A man bequeathed $50,000 to his daughter, payable on her twenty-fifth birthday, with the provision that in case she married before that time, the present worth of the bequest at the time of marriage should then be paid. If she married at 21, how much would she receive, interest being 4 % per annum and compounded quarterly?

9. What is the rate of interest if the present worth of $24,000 due in 7 years is $19,500?

10. In how many years is $5000 due if its present worth is $3500, the rate of interest being $3\frac{3}{4}$ % compounded annually?

HISTORICAL NOTE

Logarithms were given to the world full fledged on a single day when **John Napier**'s work on the subject was published in 1614. "It is one of the curiosities of the history of science that Napier constructed logarithms before exponents were used." Napier's development of his idea of logarithms is too complicated for us to follow here, but "it is so unique and so different from all other modes of presenting the subject that there cannot be any doubt that this invention is entirely his own." Napier's purpose was to shorten the labor of multiplication and division which was becoming so serious in the then rapidly developing science of Astronomy; and since his time logarithms have been one of the chief elements in making possible the tremendous amount of calculation which is necessary in modern science and in its application to the industries.

Henry Briggs (1556–1631), professor of mathematics at Oxford, improved Napier's system by making it more directly applicable to the decimal notation. Briggs constructed extensive tables to 14 decimal places.

Tables of logarithms were published in Germany in 1625, in France in 1624, and in Italy in 1626.

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